Transducers and Translation Grammars

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Finite Transducers

Finite Transducer

A finite transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

- Q is a finite set of states
- Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O$,
  I and O are input and output alphabets
- R is a finite set of rules of the form

$$pa \rightarrow qz$$

  $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $z \in O^*$
- $s \in Q$ is the start state
- $F \subseteq Q$ is a set of final states

Input Finite Automaton

Input Finite Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a finite transducer, then

$$M_I = (Q, I, R_I, s, F)$$

where

- I is the input alphabet of M and
- $R_I = \{qa \rightarrow p \, : \, qa \rightarrow px \in R, x \in O^*\}$

is the input finite automaton

Finite Transducers – Computational Step

Configuration

$$\chi \in QI^*\{|\}O^*$$

Move

If

$$r : qa \rightarrow pz \in R,$$

$$\chi = qaw|y,$$

$$\chi' = pw|yz,$$

then

$$\chi \Rightarrow \chi' \,[r]$$

Finite Transducers – Translation

Translation of a Word

M translates x into y if

$$sx| \Rightarrow^* f|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : sx| \Rightarrow^* f|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Finite Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* : (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* : (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Theorem

Both input and output languages are regular.

Finite Transducers – Example

Example

$$M = (\{s, q, f\}, \{!, a\}, R, s, \{f\})$$

where

$$R = \{1 : s! \to q, \qquad 3 : q! \to s,$$
$$2 : sa \to fa, \qquad 4 : qa \to f!a\}$$

$$s!!!a| \Rightarrow q!!a| \ [1] \Rightarrow s!a| \ [3] \Rightarrow qa| \ [1] \Rightarrow f|!a \ [4]$$

$$T(M) = \{(!^i a, a) \ : \ i \geq 0, i = 2k, k \geq 0\}$$
$$\cup \ \{(!^i a, !a) \ : \ i \geq 1, i = 2k + 1, k \geq 0\}$$

$$L_I(M) = \{!^i a \ : \ i \geq 0\} \qquad L_O(M) = \{a, !a\}$$

Deterministic Finite Transducer

M is deterministic if each rule $r \in R$ with $\text{lhs}(r) = pa$ satisfies

$$\{r\} = \{r' \in R \ : \ pa = \text{lhs}(r') \text{ or } p = \text{lhs}(r')\}$$

Example

Deterministic finite transducer

$$M = (\{f\}, \{0\}, \{f \to f0\}, f, \{f\}),$$

then

$$T(M) = \{(\varepsilon, 0^i) \ : \ i \geq 0\}$$

Pushdown Transducers

Pushdown Transducer

A pushdown transducer is a quintuple

$$M = (Q, \Sigma, R, s, F)$$

where

Q, s, F have the same meaning as in the case of finite transducers

Σ is an alphabet, $\Sigma \cap Q = \emptyset$, $\Sigma = I \cup O \cup P_D$,

I, O, P_D are input, output and pushdown alphabets,

$S \in P_D$ is the start pushdown symbol

R is a finite set of rules of the form

$$Apa \to uqv$$

$A \in P_D$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Input Pushdown Automaton

Input Pushdown Automaton

Let $M = (Q, \Sigma, R, s, F)$ be a pushdown transducer, then

$$M_I = (Q, I \cup P_D, R_I, s, F)$$

where

- I and P_D are the input and the pushdown alphabets of M
- $R_I = \{Aqa \rightarrow up : Aqa \rightarrow upv \in R, v \in O^*\}$

is the input pushdown automaton

Pushdown Transducers – Computational Step

Configuration

$$\chi \in P_D^* Q I^* \{|\} O^*$$

Move

If
$$r : Aqa \rightarrow upv \in R,$$
$$\chi = zAqaw|y,$$
$$\chi' = zupw|yv,$$
then
$$\chi \Rightarrow \chi' \; [r]$$

Pushdown Transducers – Translation

Translation of a Word

M translates x into y if

$$Ssx| \Rightarrow^* zf|y \text{ where } f \in F$$

Translation Defined by M

$$T(M) = \{(x, y) \in I^* \times O^* : Ssx| \Rightarrow^* zf|y, f \in F\}$$

- $\Rightarrow^*$ denotes the reflexive and transitive closure of $\Rightarrow$

Pushdown Transducers – Input and Output Language

Input Language

$$L_I(M) = \{x \in I^* : (x, y) \in T(M) \text{ for some } y \in O^*\}$$

Output Language

$$L_O(M) = \{y \in O^* : (x, y) \in T(M) \text{ for some } x \in I^*\}$$

Pushdown Transducers – Example

Example

$$M = (\{s, q, f\}, \{S, A, +, *, a\}, R, s, \{f\})$$

where

$$R = \{1 : Ss \rightarrow SAq, \qquad 4 : Aq* \rightarrow *AAq,$$
$$2 : Aqa \rightarrow qa, \qquad 5 : +q \rightarrow q+,$$
$$3 : Aq+ \rightarrow +AAq, \qquad 6 : *q \rightarrow q*, \qquad 7 : Sq \rightarrow f\}$$

$$Ss + *aaa| \Rightarrow SAq + *aaa| \ [1] \Rightarrow S + AAq * aaa| \ [3]$$
$$\Rightarrow S + A * AAqaaa| \ [4] \Rightarrow S + A * Aqaa|a \ [2] \Rightarrow S + A * qa|aa \ [2]$$
$$\Rightarrow S + Aqa|aa* \ [6] \Rightarrow S + q|aa * a \ [2] \Rightarrow Sq|aa * a + \ [5]$$
$$\Rightarrow f|aa * a + \ [7]$$

$$T(M) = \{(pre, post) \ : \ pre = \text{prefix expression}, \ post = \text{postfix expression}\}$$

Pushdown Transducers – Determinism

Deterministic Pushdown Transducer

M is deterministic if each rule $r \in R$ with $\mathsf{lhs}(r) = Apa$ satisfies

$$\{r\} = \{r' \in R \; : \; Apa = \mathsf{lhs}(r') \text{ or } Ap = \mathsf{lhs}(r')\}$$

Extended Pushdown Transducer

$M = (Q, \Sigma, R, s, F)$ is extended if R is a finite set of productions of the form

$$zpa \rightarrow uqv$$

$z \in P_D^*$, $p, q \in Q$, $a \in I \cup \{\varepsilon\}$, $u \in P_D^*$, $v \in O^*$

Translation Grammars

Translation Grammar

A translation grammar is a quadruple

$$G = (N, T, P, S)$$

where

N, S are defined as usual, $S \in N$

T is a terminal alphabet, $T \cap N = \emptyset$, $T = I \cup O$

I and O are input and output alphabets

P is a finite set of productions of the form

$$A \twoheadrightarrow u_0 B_1 u_1 \ldots B_n u_n | v_0 B_1 v_1 \ldots B_n v_n$$

| is a special symbol, $A \in N$, $B_i \in N$, $u_j \in I^*$, $v_j \in O^*$,
$i = 1, \ldots, n$, $j = 0, \ldots, n$

Translation Grammars – Direct Derivation

Notation

For

$$p = A \twoheadrightarrow u_0 B_1 u_1 \ldots B_n u_n | v_0 B_1 v_1 \ldots B_n v_n,$$

- $\mathrm{lhs}(p) = A$
- $\mathrm{irhs}(p) = u_0 B_1 u_1 \ldots B_n u_n$
- $\mathrm{orhs}(p) = v_0 B_1 v_1 \ldots B_n v_n$

Direct Derivation

For $G = (N, T, P, S)$, $p \in P$, $x, y, u, v \in (N \cup T)^*$, then

$$x\,\mathrm{lhs}(p)y | u\,\mathrm{lhs}(p)v \Rightarrow x\,\mathrm{irhs}(p)y | u\,\mathrm{orhs}(p)v$$

where the same number of nonterminals occurs in x and u.

Translation Grammars – Translation

Translation Defined by G

$$T(G) = \{u|v \; : \; S|S \Rightarrow^* u|v, u \in I^*, v \in O^*\}$$

Input Grammar

$$G_I = (N, I, P_I, S)$$

where $P_I = \{A \rightarrow x \; : \; A \twoheadrightarrow x|y \in P\}$

Output Grammar

$$G_O = (N, O, P_O, S)$$

where $P_O = \{A \rightarrow y \; : \; A \twoheadrightarrow x|y \in P\}$

Translation Grammars – Example

Example

Let the translation grammar G be defined by the following productions:

$$\langle expr \rangle \rightarrowtail \langle expr \rangle + \langle term \rangle | \langle expr \rangle \langle term \rangle +$$
$$\langle expr \rangle \rightarrowtail \langle term \rangle | \langle term \rangle$$
$$\langle term \rangle \rightarrowtail \langle term \rangle * \langle factor \rangle | \langle term \rangle \langle factor \rangle *$$
$$\langle term \rangle \rightarrowtail \langle factor \rangle | \langle factor \rangle$$
$$\langle factor \rangle \rightarrowtail (\langle expr \rangle) | \langle expr \rangle$$
$$\langle factor \rangle \rightarrowtail a | a$$

$$\langle expr \rangle | \langle expr \rangle \Rightarrow^* (a + a) * a | aa + a*$$

G translates an infix expression with $+$ and $*$ to the corresponding postfix expression.

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