

Lindenmayer Systems

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Modern Formal Language Theory, 2007

0L System

An **0L system** (0 stands for zero-sided context, i.e. context-free productions) is a triple

$$G = (T, P, w)$$

where

T is an alphabet

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in T$ and $x \in T^*$

w is the start string (axiom), $w \in T^+$

D0L and P0L System

D0L System

If for each $a \in T$ there is exactly one production

$$a \rightarrow x \in P,$$

then G is a **D0L system** (**D** stands for **D**eterministic)

P0L System

If for each $a \rightarrow x \in P$,

$$x \neq \varepsilon,$$

then G is a **P0L system** (**P** stands for **P**ropagating)

Direct Derivation

For some $n \geq 1$,

$$a_1 a_2 \dots a_n \Rightarrow x_1 x_2 \dots x_n$$

if for each $i = 1, \dots, n$,

$$a_i \rightarrow x_i \in P$$

Generated Language

For an L system $G = (T, P, w)$,

$$L(G) = \{y : w \Rightarrow^* y\}$$

Length set of L

$$|L| = \{|x| : x \in L\}$$

Example

$$G = (\{a, b, c\}, \{a \rightarrow abcc, b \rightarrow bcc, c \rightarrow c\}, a)$$

$$a \Rightarrow abcc \Rightarrow abccbcccc \Rightarrow abccbccccbcccccc \dots$$

$$|L(G)| = \{i^2 : i \text{ is a natural number}\}$$

0L System – Example I

Example

PD0L system $G = (\{a\}, \{a \rightarrow aa\}, a)$

$$L(G) = \{a^{2^n} : n \geq 0\}$$

Example

0L system $G = (\{a, b\}, \{a \rightarrow b, b \rightarrow ab\}, a)$

$$a \Rightarrow b \Rightarrow ab \Rightarrow bab \Rightarrow abbab \Rightarrow \dots$$

$$|L(G)| = \{i : i \geq 1, i \text{ is a Fibonacci number}\}$$

Every Fibonacci number f_n (for all $n \geq 0$) is defined as

- $f_0 = 0, f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \textcolor{red}{1})$ where P contains

$\textcolor{red}{1}$	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

1

1

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \textcolor{red}{1})$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#3

2	3
---	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \mathbf{1})$ where P contains

$\mathbf{1}$	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#4

2	2	4
---	---	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \textcolor{red}{1})$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#504

2	2	5/4
---	---	-----

(...) branch

8 branch position

0 oblique wall

vertical wall

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Example (Red Alga)

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$\textcolor{red}{1}$	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#60504

2	2	6/5	4
---	---	-----	---

(...) branch

8 branch position

0 oblique wall

vertical wall

0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \textcolor{red}{1})$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#7060504

2	2	7/6	5/4
---	---	-----	-----

(...) branch

8 branch position

0 oblique wall

vertical wall

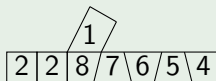
0L System – Example II

Example (Red Alga)

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1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#8(1)07060504



(...) branch

8 branch position

0 oblique wall

vertical wall

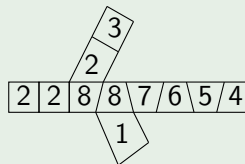
0L System – Example II

Example (Red Alga)

$G = (\{1, 2, 3, 4, 5, 6, 7, 8, (,), \#, 0\}, P, \textcolor{red}{1})$ where P contains

1	2	3	4	5	6	7	8	()	#	0
2#3	2	2#4	504	6	7	8(1)	8	()	#	0

2#2#8(2#3)08(1)07060504



(...) branch

8 branch position

0 oblique wall

vertical wall

Theorem

$\mathcal{L}(0L)$ is *not* closed under union.

Proof

$\{a\} \in \mathcal{L}(0L)$ and $\{aa\} \in \mathcal{L}(0L)$, but

$$\{a, aa\} \notin \mathcal{L}(0L)$$



0L System – Closure Properties II

Theorem

$\mathcal{L}(0L)$ is **not** closed under positive closure $(+)$.

Basic Idea

Set

$$L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

and prove that

- 1 $L \in \mathcal{L}(0L)$
- 2 $L^+ \notin \mathcal{L}(0L)$

Proof of $L \in \mathcal{L}(0L)$

Set $G = (\{a, b\}, P, aa)$ with $P = \{a \rightarrow bb, b \rightarrow bb\}$. Then,

$$L(G) = L = \{aa\} \cup \{b^{2^n} : n \geq 2\}$$

Proof of $L^+ \notin \mathcal{L}(0L)$

(Proof by contradiction.) Assume that there exists an 0L system

$$G = (\{a, b\}, P, w)$$

such that $L(G) = L^+$. As $\varepsilon \notin L^+$, G is propagating. Thus,

$$w = aa.$$

Consider $a^4 \in L^+$.

1 Assume $a^2 \Rightarrow a^4$.

a Let $\{a \rightarrow a, a \rightarrow aaa\} \subseteq P$. Then, $a^2 \Rightarrow b^4$ or $a^4 \Rightarrow b^4$. Thus,

$$a \rightarrow b^i \in P$$

for some $i \in \{1, 2, 3\}$. Hence, $aa \Rightarrow ab^i$ and $ab^i \notin L^+$ – a contradiction.

Proof of $L^+ \notin \mathcal{L}(0L)$

1 b Assume $a^2 \Rightarrow a^4$ and $a \rightarrow aa \in P$.

■ If $a^2 \Rightarrow b^4$, then

$$a \rightarrow b^i$$

for some $i \in \{1, 2, 3\}$. Thus, $a^2 b^i \in L(G)$ – a contradiction.

■ If $a^4 \Rightarrow b^4$,

$$a \rightarrow b \in P.$$

Thus, $aab \in L(G)$ – a contradiction.

c Assume $a^2 \Rightarrow b^4 \Rightarrow a^4$. Then,

$$\{a \rightarrow bb, b \rightarrow a\} \subseteq P.$$

Consider any $x \in L(G)$ with $|x| = 6$. Then, $x \in \{a^2 b^4, b^4 a^2, a^6\}$.

Proof of $L^+ \notin \mathcal{L}(0L)$

1 c $a^4 \not\Rightarrow x$.

A $a^4 \Rightarrow b^4 a^2$.

If $a \rightarrow a^i \in P$, $i \in \{1, 2\}$, then $aa \Rightarrow a^i b^2 \in L(G)$ – a contradiction.

If $b \rightarrow b \in P$, then $bbba \in L(G)$ – a contradiction.

B $a^4 \Rightarrow a^2 b^4$ – analogy.

C If $a^4 \Rightarrow a^6$, then $bba^i \in L(G)$ – a contradiction.

d $b^4 \not\Rightarrow x$

A $b^4 \Rightarrow b^4 a^2$. Then $b \rightarrow b^i \in P$ for some $i \geq 1$. Then, $b^4 \Rightarrow b^{3i} a$ – a contradiction.

B $b^4 \Rightarrow a^2 b^4$ – analogy.

C ...

e $a^2 \not\Rightarrow x$.

⋮



Theorem

$\mathcal{L}(OL)$ is *not* closed under

- *homomorphism*
- *inverse homomorphism*
- *intersection and intersection with a regular set*
- *concatenation*
- *complementation*

Theorem

$\mathcal{L}(OL)$ is closed under reversal.

Theorem

If $L \in \mathcal{L}(0L)$, $L \subseteq \{a\}^$, then $L^* \in \mathcal{L}(0L)$.*

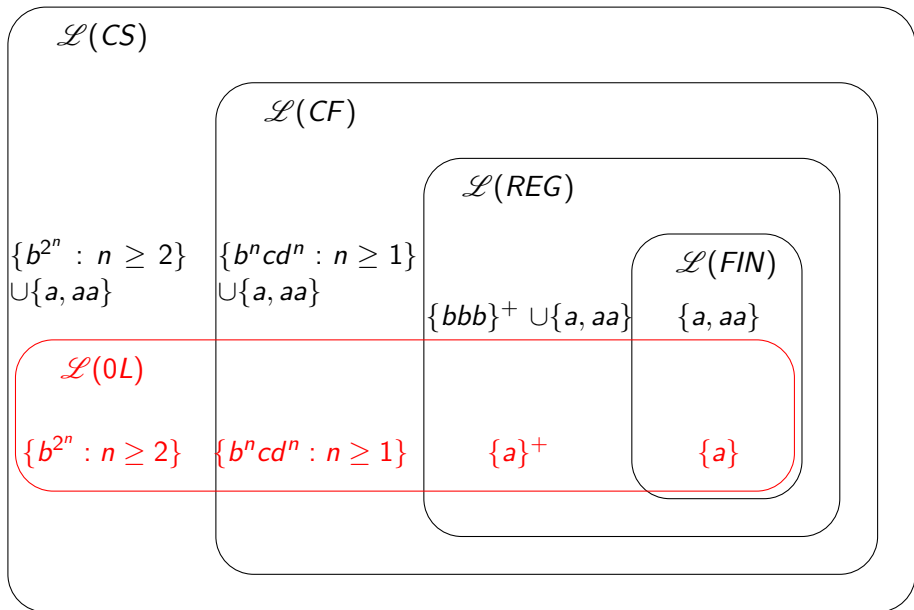
Theorem

If L is finite, then $L^ \in \mathcal{L}(0L)$.*

Theorem

If $L \in \mathcal{L}(0L)$, $L \subseteq \{a\}^$, $\varepsilon \in L$, then L is regular.*

0L Systems in Chomsky Hierarchy



E0L System

An **E0L system** is a quadruple

$$G = (V, T, P, w)$$

where

V is a total alphabet

T is a terminal alphabet, $T \subseteq V$

P is a finite set of productions of the form

$$a \rightarrow x$$

with $a \in V$ and $x \in V^*$

w is the axiom, $w \in V^+$

E0L System – Generated Language

- $\Rightarrow, \Rightarrow^*$ – by analogy with 0L systems

Generated Language

For an E0L system $G = (V, T, P, w)$,

$$L(G) = \{y \in T^* : w \Rightarrow^* y\}$$

Example

E0L system

$$G = (\{S, a, b\}, \{a, b\}, \{S \rightarrow a, S \rightarrow b, a \rightarrow aa, b \rightarrow bb\}, S)$$

$$L(G) = \{a^{2^n} : n \geq 0\} \cup \{b^{2^n} : n \geq 0\}$$

$$L(G) \in \mathcal{L}(E0L) - \mathcal{L}(0L)$$

Example

E0L system

$$G = (\{A, a, b\}, \{a, b\}, \{A \rightarrow A, A \rightarrow a, a \rightarrow aa, b \rightarrow b\}, AbA)$$

$$L(G) = \{a^{2^n}ba^{2^m} : n, m \geq 0\}$$

Theorem

$$\mathcal{L}(CF) \subset \mathcal{L}(E0L).$$

Proof

Homework

Theorem

$\mathcal{L}(E0L)$ is closed under

- union
- concatenation
- positive closure
- intersection with a regular set

Theorem

$\mathcal{L}(E0L)$ is **not** closed under inverse homomorphism.

T0L System

A **T0L system** (**T** stands for **T**ables) is an $(n + 2)$ -tuple

$$G = (T, P_1, P_2, \dots, P_n, w)$$

where

- $n \geq 1$
- for **all** $i = 1, \dots, n$, $G_i = (T, P_i, w)$ is an 0L system

Direct Derivation

For $u, v \in T^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (T, P_i, w)$ for **some** $i \in \{1, \dots, n\}$

- \Rightarrow^* , $L(G)$ – by analogy with 0L systems

ETOL System

An **ETOL system** is an $(n + 3)$ -tuple

$$G = (V, T, P_1, P_2, \dots, P_n, w)$$

where

- $n \geq 1$
- for **all** $i = 1, \dots, n$, $G_i = (V, T, P_i, w)$ is an EOL system

Direct Derivation

For $u, v \in V^*$,

$$u \Rightarrow v \text{ in } G$$

if $u \Rightarrow v$ in $G_i = (V, T, P_i, w)$ for **some** $i \in \{1, \dots, n\}$

- \Rightarrow^* , $L(G)$ – by analogy with EOL systems

Two-Table ET0L System

Theorem

For every ET0L system H , there exists an equivalent ET0L system of the form $G = (V, T, P_1, P_2, w)$.

Proof

Let

$$H = (W, T, R_1, \dots, R_n, w)$$

be an n -table ET0L system. Define the two-table ET0L system

$$G = (V, T, P_1, P_2, w)$$

with

- 1 $V = W \cup \{\langle a, i \rangle : a \in W, i = 1, \dots, n\}$
- 2 $P_1 = \{a \rightarrow \langle a, 1 \rangle : a \in W\} \cup \{\langle a, j \rangle \rightarrow \langle a, j+1 \rangle : 1 \leq j \leq n-1\}$
- 3 $P_2 = \{\langle a, k \rangle \rightarrow x : 1 \leq k \leq n, a \rightarrow x \in R_k\}$

Generative Power of E0L and ET0L Systems

Theorem

$$\mathcal{L}(CF) \subset \mathcal{L}(E0L) \subset \mathcal{L}(ET0L) \subset \mathcal{L}(CS).$$

Proof – Basic Idea

1 $\mathcal{L}(E0L) \subset \mathcal{L}(ET0L)$ can be proved by showing that

- $\{\#w\#w\#w : w \in \{a, b\}^*\}$ or
- $\{a^i b^j a^i : j \geq i \geq 1\}$

can be generated by an ET0L system and cannot be generated by any E0L system

2 $\mathcal{L}(ET0L) \subset \mathcal{L}(CS)$ can be proved by showing that

- $\{(ab^n)^m : m \geq n \geq 1\}$ or
- $\{a^{2^{2^n}} : n \geq 0\}$

are context-sensitive languages which cannot be generated by any ET0L system

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