Cryptography

Complexity Theory

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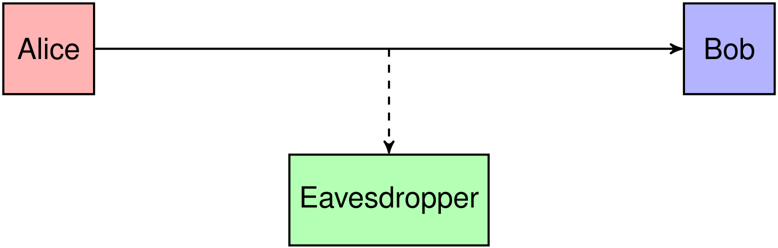
Motivation

- Hardness of problems is not always bad ...
- ...sometimes, it is a resource to be exploited!
- We wish to find problems that are quickly solvable with a partial knowledge of the solution, but very hard without it (including approximation/probabilistic algorithms).
- We will look at cryptography from the complexity's point of view. For history, side channel attacks, etc., refer to the KRY class.

Note: in this lecture we fix $\Sigma = \{0, 1\}$.

General Setting



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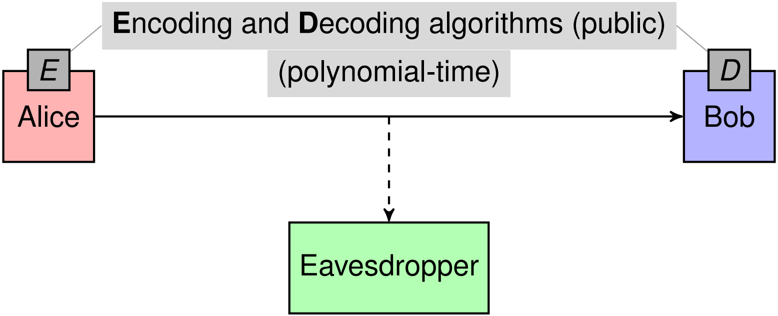

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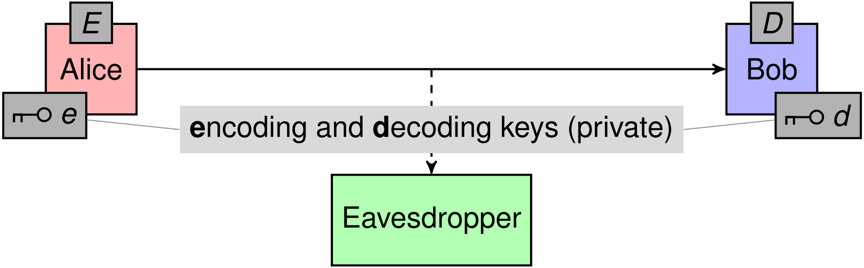

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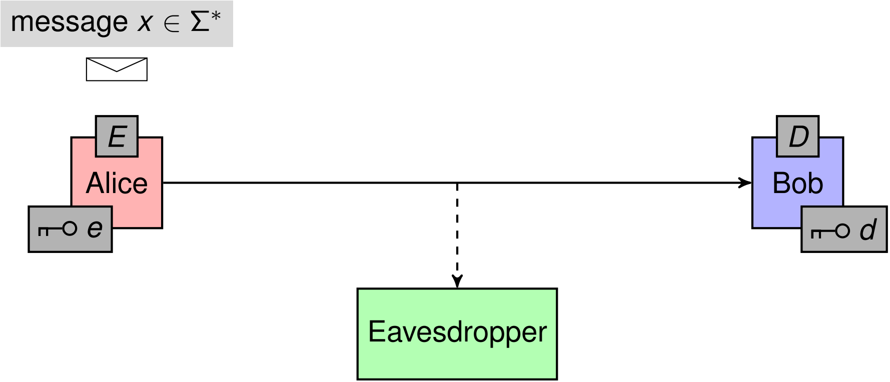

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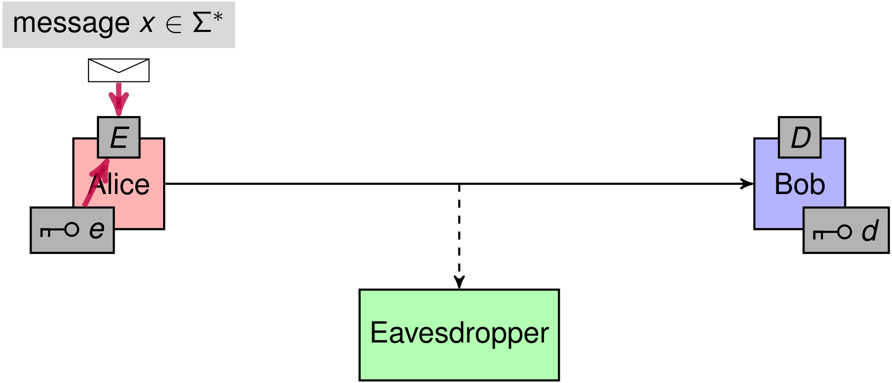

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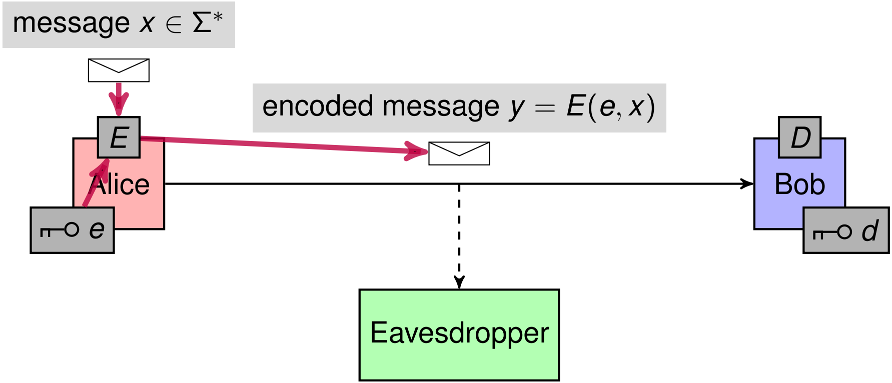

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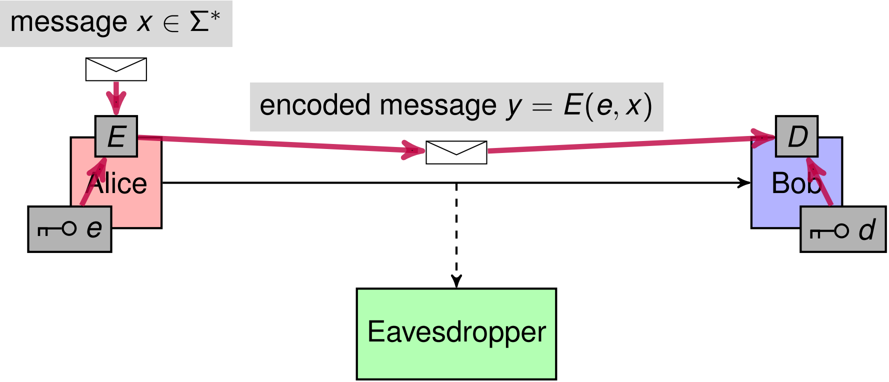

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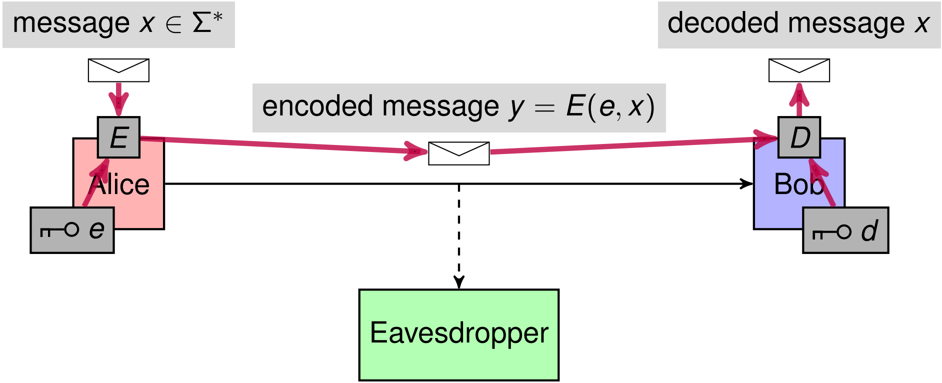

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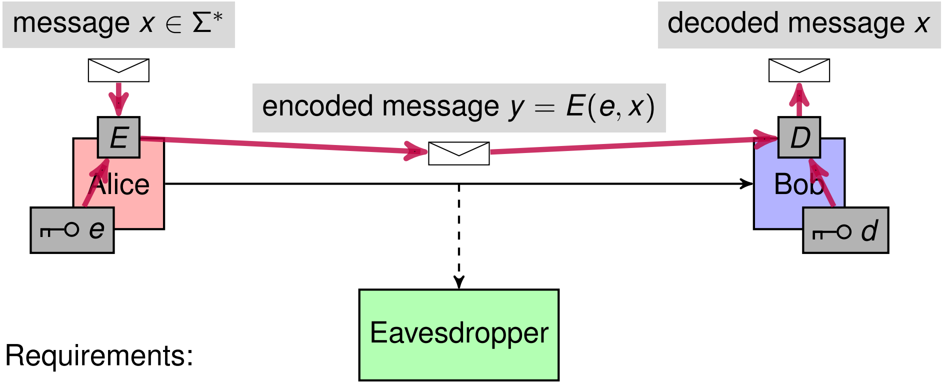

Requirements:

1. $x = D(d, E(e, x))$

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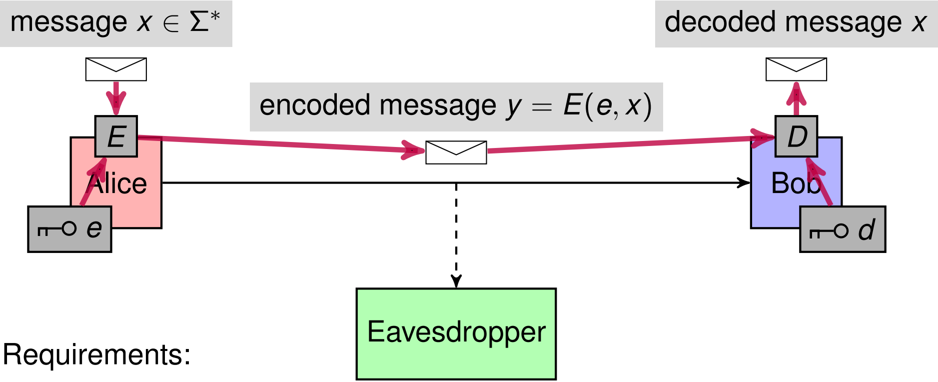

Requirements:

1. $x = D(d, E(e, x))$
2. Eavesdropper not able to compute x from y without knowing d

One-Time Pad

Example (one-time pad):

- let $e = d$ be a string $w \in \Sigma^*$ of length $|x|$ and

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   - suppose $y_1 = E(w, x_1)$, $y_2 = E(w, x_2)$
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2. distribution of keys to the parties

Public-Key Cryptography

Public-key Cryptosystem

- d — secret and private for Bob,
- e — public,
- it is computationally infeasible to deduce d from e, and x from y without knowing d

Issues:

- when guessing x, it is easy to check whether $x \stackrel{?}{=} D(d, y)$ by checking whether $y = E(e, x)$
- and since $|x| \leq |y|^k$ for some $k > 0$, compromising it is in **FNP**,
- $\implies$ public-key cryptosystems exists only if $\mathbf{P} \neq \mathbf{NP}$.

... one-way functions (inhabitants of $\mathbf{FNP} \setminus \mathbf{FP}$)

One-way Functions

A function $f : \Sigma^* \to \Sigma^*$ is one-way if:

1. f is injective and $\forall x \in \Sigma^*, |x|^{\frac{1}{k}} \leq |f(x)| \leq |x|^k$ for some $k > 0$,
2. $f \in \mathbf{FP}$,
3. $f^{-1} \notin \mathbf{FP}$ (and therefore $f^{-1} \in \mathbf{FNP} \setminus \mathbf{FP}$).

If there exist one-way functions, then $\mathbf{P} \neq \mathbf{NP}$.

RSA

The RSA function:

- Proposed by Ron **R**ivest, Adi **S**hamir, and Leonard **A**dleman.
- Uses integer multiplication and exponentiation modulo a prime.
- $p, q \quad \ldots$ two large primes (private), their product pq (public)
- $1 < e < \phi(pq) \quad \ldots$ an integer coprime with $\phi(pq)$ (public)
  - $\phi(pq) = pq(1 - \frac{1}{p})(1 - \frac{1}{q}) = pq - p - q + 1$ Euler's totient function
- $d \quad \ldots$ an integer s.t. $\quad e \cdot d \equiv 1 \mod \phi(pq)$ (private)
- $E = \lambda\, x \,.\, x^e \mod pq$
- $D = \lambda\, y \,.\, y^d \quad (= (x^e)^d = x^{e \cdot d} = x^{1+k\phi(pq)} = x \mod pq)$
  - if $1 \leq x < pq$ and x and pq are coprime, then $x^{\phi(pq)} = 1 \mod pq$ Euler's totient theorem (generalization of Fermat's little theorem)
- fast factoring can break RSA (p, q, and e can be used to get d)

UP

Definition (UP)

UP is the class of languages accepted by unambiguous polynomial-time bounded nondeterministic Turing machines.

- Unambiguous NTM: for any input there is at most 1 accepting run.
- Obviously, $\mathbf{P} \subseteq \mathbf{UP} \subseteq \mathbf{NP}$.
- It is believed that $\mathbf{UP} \neq \mathbf{NP}$.

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- Suppose there is a one-way function f.
- Consider the language $L_f = \{(x, y) \mid \exists z \in \Sigma^* \, . \, f(z) = y \wedge z \leq x\}$.
  (words over Σ ordered first by length and then lexicographically)

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  f being injective implies this happens at most once
- $L_f \notin$ **P**: if there were a **PTIME** algorithm for L_f, we could invert f in **PTIME** using binary search $\implies$ f would not be one-way
- therefore, **P** $\subset$ **UP** (because $L_f \in$ **UP** $\setminus$ **P**)

UP

"$\Rightarrow$":

- Suppose there is a language $L \in \textbf{UP} \setminus \textbf{P}$.
- Let U be an unambiguous TM accepting L.
- Let x be an encoding of an accepting computation of U on input y.
- Define $f_U(x) = 1y$ and $f_U(z) = 0z$ if z is not such an encoding.
- f_U is one-way, because
  - f_U is well-defined (y can be "read off" x in **PTIME**),
  - lengths of x and $f_U(x)$ are polynomially related,
  - f_U is injective ($f(x) = f(x') \implies x = x'$),
  - inverting f_U in **PTIME** would imply $L \in \textbf{P}$. $\square$

One-way Functions Revisited

Worst-case performance of algorithms

- not a good concept for cryptography!
- hard problems need to be densely populating the problem space,
- we need to refine the requirement for one-way functions:

> 3 $f^{-1} \notin$ **FP** (and therefore $f^{-1} \in$ **FNP** $\setminus$ **FP**).

to a stronger requirement:

> 3 there is no $k \in \mathbb{N}$, and no algorithm which, for large enough n, in time $\mathcal{O}(n^k)$ successfully computes $f^{-1}(y)$ for at least $\frac{2^n}{n^k}$ strings of length n.

- i.e. there is no **PTIME** algorithm that successfully inverts f on a polynomial fraction of the inputs of length n.

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- Note: any message can be split into bits and send using this scheme. This avoids the problems of repetition, guessing messages, etc.

Protocols

Signature

- modification of a document that unmistakably identifies the sender,
- commutative public-key cryptosystems can be exploited:
- Alice sends a signed message

$E(e) \circ D(d) = D(d) \circ E(e) = \mathrm{id}$

$$S_{Alice}(x) = (x, D(d_{Alice}, x))$$

private

  - i.e. Alice sends the original message with its decoded counterpart
- given a signed message (x, s) anyone can check whether

$$E(e_{Alice}, s) = x$$

public

  - i.e. check that the signature is valid
- the RSA cryptosystem can be used.

Protocols

Mental Poker

- 3 n-bit numbers $a < b < c$ (cards)
- Alice and Bob to randomly choose one card each, such that:
  1. their cards are different,
  2. all 6 allowed outcomes have the same probability,
  3. Alice's (B's) card is known only to Alice (B) until she announces it,
  4. the outcome is indisputable.
- The person with the highest number wins.

Protocols

Mental Poker — a solution:

1. The players agree on a single large prime number p.
2. Each player generates two secret keys:
   - an encryption key e_{Alice}, e_{Bob},
   - a decryption key d_{Alice}, d_{Bob},
   - such that $e_{Alice}d_{Alice} = e_{Bob}d_{Bob} = 1 \mod p - 1$.
3. Alice encrypts and sends to Bob $a^{e_{Alice}}, b^{e_{Alice}}, c^{e_{Alice}} \pmod p$.
4. Bob randomly chooses one message, say $b^{e_{Alice}}$, and returns it to Alice to be her card (Alice decodes it with d_{Alice} to obtain b).
5. Bob encrypts and sends to Alice $a^{e_{Alice}e_{Bob}}, c^{e_{Alice}e_{Bob}} \pmod p$.
6. Alice randomly chooses one message, say $a^{e_{Alice}e_{Bob}}$, decodes it with d_{Alice} and sends $a^{e_{Bob}} \mod p$ to Bob as his card.

Protocols

Zero Knowledge Example: consider the problem of 3-COLOURING of a graph $G = (V, E)$. Suppose Alice knows the colouring $\chi : V \to \{00, 11, 01\}$ and wants to persuade Bob of the fact, without revealing χ to him.

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A multiple round protocol, where in each step

1. Alice generates a random permutation π of the 3 colours.
2. Then she generates an RSA key pair (p_i, q_i, d_i, e_i) for each $i \in V$.
3. For every $i \in V$ she computes the probabilistic encoding (y_i, y_i'), according to the i-th RSA system, of i's new colour $b_i b_i' = \pi(\chi(i))$
4. For every $i \in V$ she sends $(e_i, p_i q_i, y_i, y_i')$ to Bob.
5. Now, Bob picks a random edge $(k, l) \in E$ and Alice reveals the secret keys d_k and d_l of the endpoints.
6. Bob computes $b_k b_k'$ and $b_l b_l'$ and checks that indeed $b_k b_k' \neq b_l b_l'$.

Protocols

- If Alice does not have a legal colouring, then the probability of finding an edge $(k, l) \in E$, s.t. $b_k b'_k = b_l b'_l$, is at least $\frac{1}{|E|}$.
- After $n|E|$ rounds, the probability of Bob finding out Alice has no legal colouring is at least $1 - e^{-n}$.
- But if Alice has a legal colouring, Bob has not learned anything about it.