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FORMAL CONCEPT ANALYSIS WITH GRADED AFFIRMATIONS AND DENIALS

FORMÁLNÍ KONCEPTUÁLNÍ ANALÝZA S POTVRZENÍMI A ZAPŘENÍMI VE STUPNÍCH

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Preface

Formal Concept Analysis (FCA) is a method of analysis of relational data which has proved to be useful in many areas of computer science. In its basic setting FCA is one-valued: it works only with affirmations that objects have attributes. If a user needs to express a denial of incidence, i.e. that an object does not have an attribute, he can easily achieve it using a logical negation. This is no longer the case for graded settings, where the affirmations and denials of incidences between objects and attributes are a matter of degrees. Management of graded affirmations is well elaborated in the literature because it represents a direct generalization of a one-valued character of FCA. In contrast, graded denials have received little attention. This habilitation thesis provides a thoroughly elaborated framework for handling data with graded denials and data with both graded denials and graded affirmations in FCA. A special attention is given to structures behind FCA in a graded setting.

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Contents

1 Introduction

The need to extract potentially useful information from an ever-growing amount of available data is generally recognized by both academia and business. The extracted information usually comes in the form of a reasonably small number of understandable patterns such as clusters, if-then rules (association rules, functional dependencies), etc. The process of such extraction is called Knowledge Discovery in Databases (KDD). Many KDD methods and techniques have been developed in the past few decades; one being Formal Concept Analysis (FCA) [29, 23]. Its core notion, formal concept, is a mathematical formalization of a traditional view of conceptual knowledge. As people naturally reason about reality in terms of concepts the patterns delivered by FCA are easy to understand and interpret.

Formal Concept Analysis is a method of knowledge representation, information management and data analysis invented by Rudolf Wille. Solid mathematical and computational foundations of FCA were developed in the 1980s. In the past two decades or so, FCA has enjoyed considerable interest in various communities. Many papers on applications of FCA in various domains have appeared, including those in premier journals and conferences. The method is based on a formalization of a certain philosophical view of conceptual knowledge which goes back to Port-Royal logic [1, 41].

Some of the most interesting applications of FCA are arguably in computer science. It has been applied in software engineering [61, 36, 62], web mining [26, 27], organization of web search results [25, 24], text mining and linguistics [37], analysis of medical and biological data [17, 40, 39], and crime data [51, 52].

The basic input data for FCA is a flat table, called a formal context, in which rows represent objects, columns represent attributes. Each entry of the table contains a cross if the corresponding object has the corresponding attribute, and is otherwise left blank (Fig. 1).

The basic notion in FCA is that of a formal concept. A formal concept consists of two collections: *extent*—a collection of all objects sharing the same attributes, and *intent*—a collection of all the shared attributes.

FCA represents knowledge discovered in the input data in two ways. The first one is a *concept lattice*—a hierarchy of formal concepts present in the formal context (Fig. 2). The second one is *attribute implications*—if-then rules describing dependencies among attributes in the formal context.

	a	b	c	d	e	f	g
1	×	×		×	×	×	×
2	×	×	×				×
3				×			×
4	×	×	×				×

Figure 1: Formal context with objects $1, 2, 3, 4$ and attributes $a, b, \ldots, g$.

FCA in its basic setting deals with one-valued data; i.e. presence of an element in a formal context, in a concept, or in an attribute implication represents an affirmation, while absence represents a lack of affirmation. In particular, each cross in the formal context is seen as an affirmation of the form

> "the object x has the attribute y".

An absence of such affirmation does not generally mean that the object x does not have the attribute y. The Port-Royal logic additionally works another object-attribute incidence—denial of the form

> "the object x does not have the attribute y".

When a denial needs to be processed by FCA, one can easily introduce a negative attribute, for example 'not y', and add the affirmation

> "the object x has the attribute 'not y'".

This way of managing denials in FCA can be found in [48, 49, 56, 57, 58, 60, 59]. We see that denials are easily handled in the basic setting of FCA with one-valued data, however this is no longer the case for graded data.

In everyday life we use concepts which are not sharply bounded (e.g. 'great dancer' or 'middle aged man'). In terms of FCA, objects and attributes need not be divided sharply by a formal concept into those to which the formal concept applies and those to which it does not. That is to say, a formal concept applies to different objects to different, possibly intermediate, degrees. For example, the concept 'middle aged man' may apply to a 45-year old person to degree 1, to a 55-year old person to degree 0.5, and to a 65-year old person to degree 0.2. There are several ways to generalize FCA by which we are able to process such indeterminacy or uncertainty [8, 9, 54, 47, 38, 22]

	extent	intent
c_1	$\varnothing$	Y
c_2	$\{2, 4\}$	$\{a, b, c, g\}$
c_3	$\{1\}$	$\{a, b, d, e, f, g\}$
c_4	$\{1, 2, 4\}$	$\{a, b, g\}$
c_5	$\{1, 3\}$	$\{d, g\}$
c_6	X	$\{g\}$

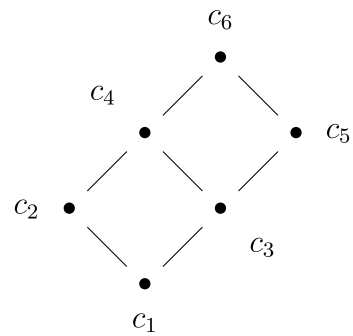


Figure 2: The formal concepts of the formal context in Fig. 1 and its concept lattice.

(see also [53] and references therein). Many of them are based on Zadeh's theory of fuzzy sets [68].

In this work, we stick with the graded setting introduced independently by Belohlavek and Pollandt [8, 9, 54] where the formal context contains truth degrees taken from a particular structure of truth degrees. Truth degree a in entry $\langle x, y \rangle$ represents an affirmation that

the object x has the attribute y *at least* to degree a.

Denials are then statements of the form:

the object x has the attribute y *at most* to degree b.

Unlike in the basic setting, here we cannot simply substitute denials by affirmations of negative attributes. The reason is that the law of double negation does not generally hold true in the graded setting. Consequently, applying negation leads to degradation of the input data.

Two main kinds of concept-forming operators, antitone (or standard) and isotone (of attribute/object-oriented), were studied [9, 30, 54, 55], compared [13, 15] and even covered under a unifying framework [10, 50]. The antitone concept-forming operators handle object-attribute incidences as affirmations, and concepts are based on sharing attributes (at least in some degree). The isotone concept-forming operators handle incidences of objects and attributes as denials, and concepts are based on the absence of the same attributes (having them at most in some degree).

The graded affirmations in FCA have been thoroughly studied in the literature while the study of graded denials is the main content of this thesis.

Contributions This thesis consists of eight selected papers whose unifying scheme is managing graded denials in FCA. They start with extensive studies of isotone concept-forming operators in FCA for graded data and lead to a general framework for FCA that handles both graded affirmations and graded denials.

The list of the papers follows. The bracketed numbers correspond to the reference numbers in the bibliography.

[43] Jan Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

[44] Jan Konecny and Michal Krupka. Block relations in formal fuzzy concept analysis. *International Journal of Approximate Reasoning*, 73:27–55, 2016.

[15] Radim Belohlavek and Jan Konecny. Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. *Information Sciences*, 199:133–137, 2012.

[3] Eduard Bartl and Jan Konecny. L-concept analysis with positive and negative attributes. *Information Sciences*, 360:96–111, 2016.

[4] Eduard Bartl and Jan Konecny. Rough fuzzy concept analysis. *Fundamenta Informaticae*, 156(2):141–168, 2017.

[44] Jan Konecny and Michal Krupka. Block relations in formal fuzzy concept analysis. *International Journal of Approximate Reasoning*, 73:27–55, 2016.

[45] Jan Konecny and Michal Krupka. Complete relations on fuzzy complete lattices. *Fuzzy Sets and Systems*, 320:64–80, 2017.

[46] Jan Konecny and Manuel Ojeda-Aciego. On homogeneous L-bonds and heterogeneous L-bonds. *International Journal of General Systems*, 45(2):160–186, 2016.

The thesis is structured as follows. Section 2 provides unified preliminaries to all the enclosed papers. It represents a brief introduction to FCA in the graded setting, [8, 9, 54]. Section 3 then contains the papers, each preceded by a short summary of its content.

2 Formal Concept Analysis for Graded Data

We introduce basic notions on complete residuated lattices, fuzzy sets and fuzzy relations and then we turn to FCA for graded data. The content of this section is not to be considered a contribution of this thesis. The only exception is the semantics of graded denials assigned to attribute-oriented concept-forming operators.

2.1 Complete Residuated Lattices

We use complete residuated lattices as basic structures of truth degrees. The truth degrees taken from these structures are used to express the strength of affirmations and denials in formal contexts and in both outputs of formal concept analysis.

A complete residuated lattice [8, 34, 64] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the partial order of $\mathbf{L}$ is denoted by $\leqslant$);
- $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
- $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

Elements of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of truth functions of "fuzzy conjunction" and "fuzzy implication." 0 and 1 denote the least and greatest elements. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Common examples of complete residuated lattices include those defined on the unit interval (i.e. $L = [0, 1]$), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \otimes c \leqslant b\}$. The three most important pairs of adjoint operations on the unit interval are

- Łukasiewicz
$$a \otimes b = \max(a + b - 1, 0),$$
$$a \rightarrow b = \min(1 - a + b, 1),$$

- Gödel

$$a \otimes b = \min(a, b),$$

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise,} \end{cases}$$

- Goguen (product)

$$a \otimes b = a \cdot b,$$

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ \frac{b}{a} & \text{otherwise.} \end{cases}$$

Instead of a unit interval we can also consider a finite chain, e.g.

$$L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}.$$

All operations on this chain are then defined analogously, see [8].

2.2 Truth-Stressing and Truth-Depressing Hedges

We endow the complete residuated lattices with additional unary operations—truth-stressing and truth-depressing hedges. These operations will serve as parameters for semantics of concept-forming operators as well as for semantics of attribute implications.

Truth-stressing hedges were studied from the point of fuzzy logic as logical connectives 'very true', see [35]. Our approach is close to that in [35]. A *truth-stressing hedge* is a mapping $^* : L \to L$ satisfying

$$1^* = 1, \quad a^* \leqslant a, \quad a \leqslant b \text{ implies } a^* \leqslant b^*, \quad a^{**} = a^* \tag{1}$$

for each $a, b \in L$.

On every complete residuated lattice $\mathbf{L}$, there are two important truth-stressing hedges:

(i) identity, i.e. $a^* = a$ $(a \in L)$;

(ii) globalization, i.e.

$$a^* = \begin{cases} 1, & \text{if } a = 1, \\ 0, & \text{otherwise.} \end{cases}$$

A *truth-depressing hedge* is a mapping ${}^{\square} : L \to L$ such that following conditions are satisfied

$$0^{\square} = 0, \quad a \leqslant a^{\square}, \quad a \leqslant b \text{ implies } a^{\square} \leqslant b^{\square}, \quad a^{\square\square} = a^{\square} \tag{2}$$

for each $a, b \in L$.

A truth-depressing hedge is a truth function of logical connective 'slightly true', see [63]. In [63] a stricter definition of the truth-depressing hedge with a connection to truth-stressing hedges is given. For our purposes, it is enough to assume conditions (2).

On every complete residuated lattice $\mathbf{L}$, there are two important truth-depressing hedges:

(i) identity, i.e. $a^{\square} = a$ $(a \in L)$;

(ii) antiglobalization, i.e.

$$a^{\square} = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let $\bullet : L \to L$ be a truth-stressing hedge or truth-depressing hedge. By $\mathrm{fix}\,(\bullet)$ we denote a set of truth degrees $a \in L$ with $a = a^{\bullet}$; that is

$$\mathrm{fix}\,(\bullet) = \{a \in L \mid a = a^{\bullet}\}.$$

2.3 L-sets and L-relations

In the basic setting, a formal concept is given by two sets—an extent which contains objects covered by the concept, and an intent which contains attributes covered by the concept. In the graded setting, the presence of objects and attributes in extents and intents is a matter of degree. We model the extents and intents using **L**-sets. Similarly, incidences between objects and attributes in the input context are a matter of degree and we model them using **L**-relations.

An **L**-*set* [32, 31] A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$. The set of all **L**-sets in a universe X is denoted L^X.

An **L**-set $A \in L^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \ldots, {}^{A(x_n)}/x_n\}.$$

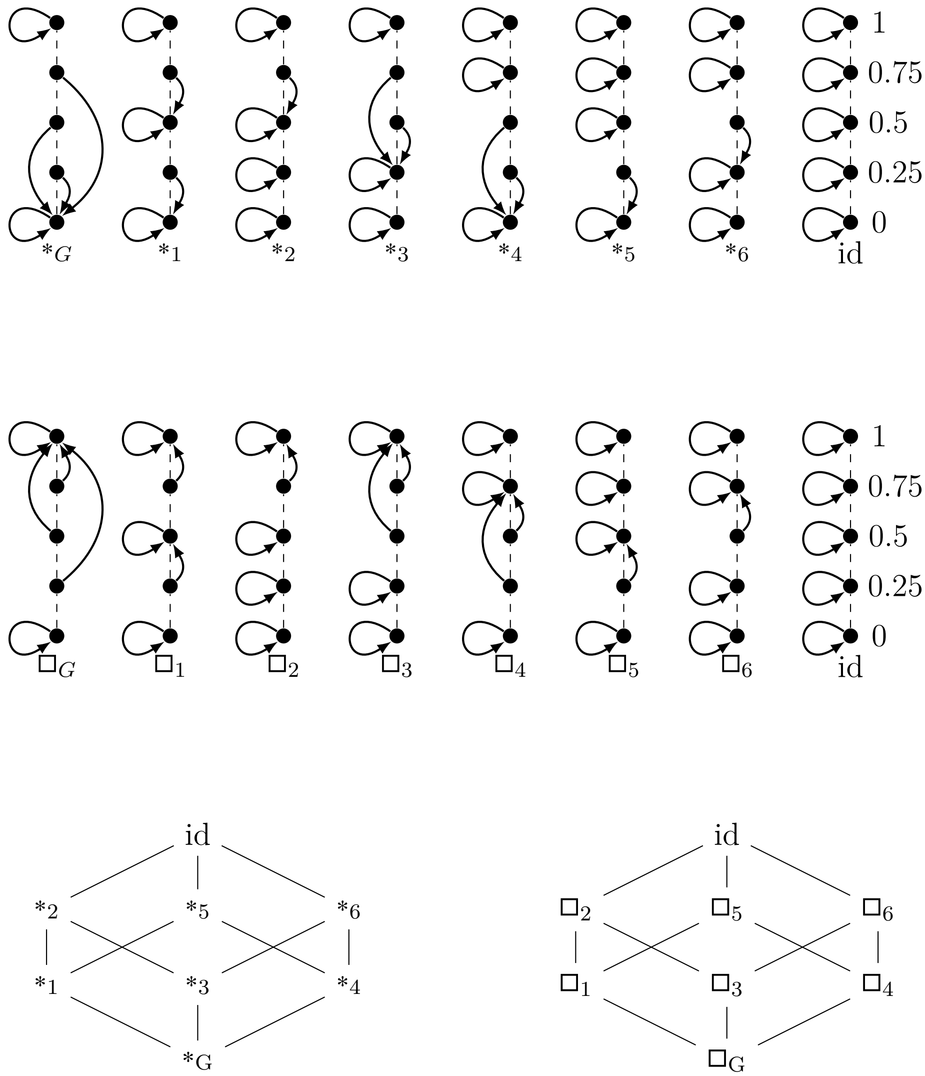


Figure 3: Truth-stressing hedges (top) and truth-depressing hedges (middle) on a five element chain and their ordering w.r.t. $\mathrm{fix}\,(\cdot) \subseteq \mathrm{fix}\,(\cdot)$ (bottom).

If there is exactly one $x \in X$ s.t. $A(x) > 0$ (i.e. $A = \{{}^{A(x)}/x\}$) we call A a singleton.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, for $a \in L$ and $A \in L^X$ we define $\mathbf{L}$-sets $a \to A$ and $a \otimes A$ in X by $(a \to A)(x) = a \to A(x)$ and $(a \otimes A)(x) = a \otimes A(x)$ for all $x \in X$ respectively. The intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$. Similarly, this is utilized for the union of $\mathbf{L}$-sets.

Additionally, for $a \in L$ and an $\mathbf{L}$-set $B \in L^X$ we define left a-multiplication $a \otimes B$ left a-shift $a \to B$ and a-complement $B \to a$ respectively by

$$
\begin{aligned}
(a \otimes B)(x) &= a \otimes B(x)\\
(a \to B)(x) &= a \to B(x)\\
(B \to a)(x) &= B(x) \to a
\end{aligned}
$$

for all $x \in X$.

Intersection and union of two $\mathbf{L}$-sets can be generalized to any number of $\mathbf{L}$-sets and even to $\mathbf{L}$-sets of $\mathbf{L}$-sets. For an $\mathbf{L}$-set $U \colon L^X \to L$, the intersection $\bigcap U$ and union $\bigcup U$ of U are $\mathbf{L}$-sets in X, defined by

$$\bigcap U(x) = \bigwedge_{A \in L^X} U(A) \to A(x), \tag{3}$$

$$\bigcup U(x) = \bigvee_{A \in L^X} U(A) \otimes A(x), \tag{4}$$

for any $x \in X$.

An $\mathbf{L}$-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp set A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$.

For $A, B \in L^X$ we define the *degree of inclusion of A in B* by

$$S(A, B) = \bigwedge_{x \in X} (A(x) \to B(x)). \tag{5}$$

The degree of inclusion generalizes the classical inclusion relation. Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, we have $A \subseteq B$ iff $A(x) \leqslant B(x)$ for each $x \in X$. Further, we set

$$A \approx^X B = S(A, B) \wedge S(B, A). \tag{6}$$

The value $A \approx^X B$ is interpreted as the *degree to which the sets A and B are similar*.

A *binary* **L**-*relation* (binary fuzzy relation) between X and Y can be thought of as an **L**-set in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). In the case $X = Y$ we call such **L**-relation also an **L**-relation on X.

A binary **L**-relation R on a set X is called *reflexive* if $R(x, x) = 1$ for any $x \in X$, *symmetric* if $R(x, y) = R(y, x)$ for any $x, y \in X$, and *transitive* if $R(x, y) \otimes R(y, z) \leqslant R(x, z)$ for any $x, y, z \in X$. R is called an **L**-*tolerance*, if it is reflexive and symmetric, **L**-*equivalence* if it is reflexive, symmetric and transitive. If R is an **L**-equivalence such that for any $x, y \in X$ from $R(x, y) = 1$ it follows $x = y$, then R is called an **L**-*equality* on X. **L**-equalities are often denoted by $\approx$. The similarity $\approx^X$ of **L**-sets (6) is an **L**-equality on L^X.

Let $\sim$ be an **L**-equivalence on X. We say that an **L**-set A in X is *compatible with* $\sim$ (or *extensional w.r.t.* $\sim$, if for any $x, x' \in X$ it holds

$$A(x) \otimes (x \sim x') \leqslant A(x'). \tag{7}$$

A binary **L**-relation R on X is *compatible with* $\sim$, if for each $x, x', y, y' \in X$,

$$R(x, y) \otimes (x \sim x') \otimes (y \sim y') \leqslant R(x', y'). \tag{8}$$

Composition Operators We use three composition operators, $\circ$, $\triangleleft$, and $\triangleright$, and consider the corresponding compositions $I = A \circ B$, $I = A \triangleleft B$, and $I = A \triangleright B$ (for $I \in L^{X \times Y}, A \in L^{X \times F}, B \in L^{F \times Y}$). In the compositions, $I(x, y)$ is interpreted as the degree to which the object x has the attribute y; $A(x, f)$ as the degree to which the factor f applies to the object x; $B(f, y)$ as the degree to which the attribute y is a manifestation (one of possibly several manifestations) of the factor f. The composition operators are defined by

$$(A \circ B)(x, y) = \bigvee_{f \in F} A(x, f) \otimes B(f, y), \tag{9}$$

$$(A \triangleleft B)(x, y) = \bigwedge_{f \in F} A(x, f) \to B(f, y), \tag{10}$$

$$(A \triangleright B)(x, y) = \bigwedge_{f \in F} B(f, y) \to A(x, z). \tag{11}$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [42]. They have natural verbal descriptions. For instance, $(A \circ B)(x,y)$ is the truth degree of the proposition "there is factor f such that f applies to object x and attribute y is a manifestation of f"; $(A \triangleleft B)(x,y)$ is the truth degree of "for every factor f, if f applies to object x then attribute y is a manifestation of f". Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known composition of binary relations.

Theorem 1 ([42, 8], associativity and distributivity of composition operators)**.** *We have*

$$R \circ (S \circ T) = (R \circ S) \circ T, \tag{12}$$
$$R \triangleleft (S \triangleright T) = (R \triangleleft S) \triangleright T, \tag{13}$$
$$R \triangleleft (S \triangleleft T) = (R \circ S) \triangleleft T, \tag{14}$$
$$R \triangleright (S \circ T) = (R \triangleright S) \triangleright T. \tag{15}$$

Furthermore, we have that

$$(\bigcup_i R_i) \circ S = \bigcup_i (R_i \circ S), \quad \textit{and} \quad R \circ (\bigcup_i S_i) = \bigcup_i (R \circ S_i), \tag{16}$$
$$(\bigcap_i R_i) \triangleright S = \bigcap_i (R_i \triangleright S), \quad \textit{and} \quad R \triangleright (\bigcup_i S_i) = \bigcap_i (R \triangleright S_i), \tag{17}$$
$$(\bigcup_i R_i) \triangleleft S = \bigcap_i (R_i \triangleleft S), \quad \textit{and} \quad R \triangleleft (\bigcap_i S_i) = \bigcap_i (R \triangleleft S_i). \tag{18}$$

Remark 1. *In [10] it is shown that $\circ$, $\triangleright$, and $\triangleleft$ can be considered to be the same composition as it can be covered by a general framework. We do not use the general framework in this thesis because most results contained here use specific properties of compositions defined by* (9),(10), *and* (11)*.*

2.4 L-Galois Connections, L-closures and L-interiors

Now we introduce the fundamental mappings behind FCA in the graded setting, specifically antitone and isotone **L**-Galois connections and **L**-closure and **L**-interior operators.

An *antitone* **L**-*Galois connection* [5] between the sets X and Y is a pair $\langle f, g \rangle$ of mappings $f : L^X \to L^Y$, $g : L^Y \to L^X$, satisfying

$$S(A_1, A_2) \leqslant S(f(A_2), f(A_1)) \qquad S(B_1, B_2) \leqslant S(g(A_2), g(B_1)) \tag{19}$$
$$A \subseteq g(f(A)) \qquad B \subseteq f(g(B)) \tag{20}$$

for every $A, A_1, A_2 \in L^X, A, A_1, A_2 \in L^Y$.

An *isotone* **L**-*Galois connection* [30] between the sets X and Y is a pair $\langle {}^\cap, {}^\cup \rangle$ of mappings ${}^\cap : L^X \to L^Y$, ${}^\cup : L^Y \to L^X$, satisfying

$$S(A_1, A_2) \leqslant S(f(A_1), f(A_2)) \qquad S(B_1, B_2) \leqslant S(g(A_1), g(B_2)) \tag{21}$$
$$A \subseteq g(f(A)) \qquad B \supseteq f(g(B)) \tag{22}$$

for every $A, A_1, A_2 \in L^X, A, A_1, A_2 \in L^Y$.

The following theorem summarizes properties of both antitone and isotone Galois connections.

Theorem 2 ([5, 30])**.** *An antitone* **L**-*Galois connection* $\langle f, a \rangle$ *satisfies the following properties:*

(i) $A_1 \subseteq A_2$ *implies* $f(A_2) \subseteq f(A_1)$ *and* $B_1 \subseteq B_2$ *implies* $g(B_2) \subseteq g(B_1)$

(ii) $S(A, g(B)) = S(B, f(A))$

(iii) $f\left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I} f(A_i)$ *and* $g\left(\bigcup_{i\in I} B_i\right)^{\downarrow} = \bigcap_{i\in I} g(B_i)$

(iv) $f(g(f(A))) = f(A)$ *and* $g(f(g(B))) = g(B)$

for each $A, A_i \in L^X, B, B_i \in L^Y$.

An isotone **L**-*Galois connection* $\langle f, g \rangle$ *satisfies the following properties:*

(i) $A_1 \subseteq A_2$ *implies* $f(A_1) \subseteq f(A_2)$ *and* $B_1 \subseteq B_2$ *implies* $g(B_1) \subseteq g(B_2)$

(ii) $S(A, g(B)) = S(f(A), B)$

(iii) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$ *and* $g\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} g(B_i)$

(iv) $f(g(f(A))) = f(A)$ *and* $g(f(g(B))) = g(B)$

for each $A, A_i \in L^X, B, B_i \in L^Y$.

Definition 1. [11, 6] A system of **L**-sets $V \subseteq L^X$ is called an **L**-*interior system* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $A \in V$ we have that $a \otimes A \in V$;

- V is closed under union, i.e. for $A_j \in V$ $(j \in J)$ we have that $\bigcup_{j\in J} A_j \in V$.

$V \subseteq L^X$ is called an $\mathbf{L}$-*closure system* if

- V is closed under $\rightarrow$-shifts, i.e. for every $a \in L$ and $A \in V$ we have that $a \rightarrow A \in V$;
- V is closed under intersection, i.e. for $A_j \in V$ $(j \in J)$ we have that $\bigcap_{j\in J} A_j \in V$.

Theorem 3. *If $\langle f, g\rangle$ an antitone $\mathbf{L}$-Galois connection between sets X and Y, then the composition $f\circ g$ is an $\mathbf{L}$-closure system on X and the composition $g \circ f$ in an $\mathbf{L}$-closure system on Y.*

If $\langle f, g\rangle$ an isotone $\mathbf{L}$-Galois connection between sets X and Y, then the composition $f \circ g$ is an $\mathbf{L}$-closure system on X and the composition $g \circ f$ in an $\mathbf{L}$-interior system on Y.

2.5 L-ordered Sets

The set of all formal concepts in the graded setting with particular $\mathbf{L}$-order forms a structure called $\mathbf{L}$-ordered set. This structure is described in this section.

An $\mathbf{L}$-*order* on a set U with an $\mathbf{L}$-equality $\approx$ is a binary $\mathbf{L}$-relation $\preceq$ on U which is compatible with $\approx$, reflexive, transitive and satisfies $(u \preceq v) \wedge (v \preceq u) \leqslant u \approx v$ for any $u, v \in U$ (antisymmetry). The tuple $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is called an $\mathbf{L}$-ordered set [8, 9]. An immediate consequence of the definition is that for any $u, v \in U$ it holds

$$u \approx v = (u \preceq v) \wedge (v \preceq u). \tag{23}$$

If $\mathbf{U} = \langle\langle U, \approx\rangle, \preceq\rangle$ is an $\mathbf{L}$-ordered set then the tuple $\langle U, {}^1\!\preceq\rangle$, where ${}^1\!\preceq$ is the 1-cut of $\preceq$, is a (partially) ordered set. We sometimes write $\leqslant$ instead of ${}^1\!\preceq$ and use the symbols $\wedge$, $\bigwedge$ resp. $\vee$, $\bigvee$ for denoting infima resp. suprema in $\langle U, {}^1\!\preceq\rangle$.

For two $\mathbf{L}$-ordered sets $\mathbf{U} = \langle\langle U, \approx_U\rangle, \preceq_U\rangle$ and $\mathbf{V} = \langle\langle V, \approx_V\rangle, \preceq_V\rangle$, a mapping $f : U \rightarrow V$ is isotone, if $(u_1 \preceq_U u_2) \leqslant (f(u_1) \preceq_V f(u_2))$, and an embedding, if $(u_1 \preceq_U u_2) = (f(u_1) \preceq_V f(u_2))$, for any $u_1, u_2 \in V$.

A mapping $f: U \to V$ is called an isomorphism of $\mathbf{U}$ and $\mathbf{V}$, if it is both, a bijection and an embedding. $\mathbf{U}$ and $\mathbf{V}$ are then called isomorphic.

An antitone mapping and dual embedding are defined by $(u_1 \preceq_U u_2) \leqslant (f(u_2) \preceq_V f(u_1))$ and $(u_1 \preceq_U u_2) = (f(u_2) \preceq_V f(u_1))$, respectively. A dual isomorphism is a bijection which is a dual embedding.

Let $\mathbf{U}$ be an $\mathbf{L}$-ordered set. For any $W \in L^U$ and $w \in U$ we set

$$\mathcal{L}W(w) = \bigwedge_{u \in U} W(u) \to (w \preceq u), \quad \mathcal{U}W(w) = \bigwedge_{u \in U} W(u) \to (u \preceq w). \tag{24}$$

The right-hand side of the first equation is the degree of "For each $u \in U$, if u is in W, then w is less than or equal to u", and similarly for the second equation. Thus, $\mathcal{L}W(w)$ ($\mathcal{U}W(w)$) can be seen as the degree to which w is less (greater) than or equal to each element of W. The $\mathbf{L}$-set $\mathcal{L}W$ (resp. $\mathcal{U}W$) is called the lower cone (resp. the upper cone) of W.

For $u, v \in U$, $u \leqslant v$, the $\mathbf{L}$-set $[\![u, v]\!] = \mathcal{U}\{u\} \cap \mathcal{L}\{v\}$ is called an $\mathbf{L}$-interval with bounds u and v. We have

$$[\![u, v]\!](w) = (u \preceq w) \wedge (w \preceq v). \tag{25}$$

Let $\mathbf{U}$ be an $\mathbf{L}$-ordered set. For any $\mathbf{L}$-set $W \in L^U$ there exists at most one element $u \in U$ such that $\mathcal{L}W(u) \wedge \mathcal{U}(\mathcal{L}W)(u) = 1$ (resp. $\mathcal{U}W(u) \wedge \mathcal{L}(\mathcal{U}W)(u) = 1$) [9, 8]. If there is such an element, we call it the infimum of W (resp. the supremum of W) and denote $\inf W$ (resp. $\sup W$); otherwise we say that the infimum (resp. supremum) does not exist.

$\mathbf{U}$ is called completely lattice $\mathbf{L}$-ordered, if for each $W \in L^U$, both $\inf W$ and $\sup W$ exist.

An important example of a completely lattice $\mathbf{L}$-ordered set is the tuple $\mathbf{L}^X = \langle\langle L^X, \approx^X\rangle, S\rangle$, where X is an arbitrary set and $\approx^X$ and S are given by (6) and (5), respectively. Infima and suprema in $\mathbf{L}^X$ are intersections and unions: for any $M \in L^{L^X}$ we have

$$\inf M = \bigcap M, \qquad \sup M = \bigcup M. \tag{26}$$

2.6 Formal L-Concept Analysis

As we have now introduced all essential mathematical notions, we can finally turn our attention to the formal $\mathbf{L}$-concept analysis. Many ways to generalize FCA can be found in the literature [8, 9, 54, 47, 38, 22] (see also [53] and references therein). From here to the end of Section 2 we present the approach of Belohlavek and Pollandt [8, 9, 54].

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary nonempty) sets and $I \in L^{X\times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read:

> "the object x has the attribute y *at least* to degree a"

or

> "the object x has the attribute y *at most* to degree a"

depending on whether the incidence between x and y is seen as an affirmation or denial.

We consider the following pairs of operators, called concept-forming operators, induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ of standard concept-forming operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ is defined, for all $A \in L^X$ and $B \in L^Y$, by

$$\begin{aligned} A^{\uparrow}(y) &= \bigwedge_{x\in X}(A(x) \to I(x, y)), \\ B^{\downarrow}(x) &= \bigwedge_{y\in Y}(B(y) \to I(x, y)). \end{aligned} \tag{27}$$

In words, the operator ${}^{\uparrow}$ assign to an **L**-set A of objects the **L**-set $A^{\uparrow}$ of attributes which are shared by all the objects in A. Analogously, the operator ${}^{\downarrow}$ assign to an **L**-set B of attributes the **L**-set $B^{\downarrow}$ of objects which have all the attributes in B.

Second, the pair $\langle{}^{\cap}, {}^{\cup}\rangle$ of attribute-oriented concept-forming operators ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ is defined by

$$\begin{aligned} A^{\cap}(y) &= \bigvee_{x\in X}(A(x) \otimes I(x, y)), \\ B^{\cup}(x) &= \bigwedge_{y\in Y}(I(x, y) \to B(y)). \end{aligned} \tag{28}$$

In words, the operator ${}^{\cap}$ assign to an **L**-set A of objects the **L**-set $A^{\cap}$ of attributes which at least one object in A has. The operator ${}^{\cup}$ assign to an

	α	β	γ
A	0.5	0	1
B	1	0.5	1
C	0	0.5	0.5
D	0.5	0.5	1

Figure 4: Example of **L**-context with objects A,B,C,D and attributes α, β, γ; **L** is a chain $0 < 0.5 < 1$ with Łukasiewicz operations.

L-set B of attributes the **L**-set $B^{\cup}$ of objects which have no other attributes than those in B.

Additionally, dual operators to attribute-oriented concept-forming will be sometimes considered. Specifically, a pair of operators ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$

$$
\begin{aligned}
A^{\wedge}(y) &= \bigwedge_{x \in X} (I(x,y) \to A(x)), \\
B^{\vee}(x) &= \bigvee_{y \in Y} (B(y) \otimes I(x,y)).
\end{aligned} \tag{29}
$$

The operators $\langle {}^{\wedge}, {}^{\vee} \rangle$ are called object-oriented concept-forming operators.

When we need to emphasize which **L**-relation induces the concept-forming operators, we use an additional subscript; for example, we write ${}^{\uparrow_I}$ instead of just ${}^{\uparrow}$.

Example 1. Consider the data (**L**-context) in Fig. 4, the objects represent employees, and the attributes represent skills.

(a) One can handle the incidences in the **L**-context as affirmations and form concepts based on having the same skills at least in some degree; such concepts are standard concepts formed by $\langle \uparrow, \downarrow \rangle$. Extents of the concepts can be interpreted as collections of employees able to fulfill a task which requires particular skill set. For example, the collection of employees able to fulfill a task which requires the skill α in full degree and the skill β in half degree can be found as $\{\alpha, {}^{0.5}/\beta\}^{\downarrow}$.

(b) Or he can handle the incidences as denials and form concept based on having the same skills at most in some degree; such concepts are are standard concepts formed by isotone concept-forming operators $\langle \cap, \cup \rangle$. Extents of the concepts can be interpreted as collections of employees who lack the same skills and need some training in them. For example, the collection of employees who lack the skill α and have the skill β at most in degree is can be found as $\{{}^{0.5}/\beta, \gamma\}^{\cup}$.

Remark 2. *Notice that the three pairs of concept-forming operators can be interpreted as compositions relations. Applying the isomorphisms $L^{1\times X} \cong L^X$ and $L^{Y\times 1} \cong L^Y$ whenever necessary, one could write them, alternatively, as follows:*

$$\begin{array}{lll} A^{\uparrow} = A \triangleleft I & A^{\cap} = A \circ I & A^{\wedge} = A \triangleright I \\ B^{\downarrow} = I \triangleright B & B^{\cup} = I \triangleleft B & B^{\vee} = I \circ B \end{array}$$

The concept-forming operators induced by **L**-contexts are in correspondence with an antitone and isotone **L**-Galois connection:

Theorem 4 ([5]). *Let $\langle X, Y, I\rangle$ be an **L**-context, $\langle f, g\rangle$ be an antitone **L**-Galois connection between X and Y. Then*

(i) $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ *is a Galois connection.*

(ii) $I_{\langle f,g\rangle}$ *defined by*

$$I_{\langle f,g\rangle}(x, y) = f(\{1/x\})(y) \tag{30}$$

*is an **L**-relation between X and Y and we have*

(iii) $\langle f, g\rangle = \langle {}^{\uparrow_{I_{\langle f,g\rangle}}}, {}^{\downarrow_{I_{\langle f,g\rangle}}}\rangle$ *and* $I = I_{\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle}$.

Theorem 5. *Let $\langle X, Y, I\rangle$ be an **L**-context, $\langle f, g\rangle$ be an isotone **L**-Galois connection between X and Y. Then*

(i) $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ *is an isotone **L**-Galois connection.*

(ii) $I_{\langle f,g\rangle}$ *defined by*

$$I_{\langle f,g\rangle}(x, y) = f(\{1/x\})(y) \tag{31}$$

*is an **L**-relation between X and Y and we have*

(iii) $\langle f, g\rangle = \langle {}^{\cap_{I_{\langle f,g\rangle}}}, {}^{\cup_{I_{\langle f,g\rangle}}}\rangle$ *and* $I = I_{\langle {}^{\cap_I}, {}^{\cup_I}\rangle}$.

Remark 3.

(a) *The standard concept-forming operators represent a direct generalization of the concept-forming operators in the ordinary setting and they become the concept-forming operators in the ordinary setting when* $\mathbf{L} = \mathbf{2}$.

(b) *The two additional pairs of concept-forming operators are not separately studied in the crisp setting, since there they are easily convertible to the standard pair of concept-forming operators due to the double negation law.*

(c) *For an* $\mathbf{L}$*-set* $A \in L^X$*, the truth degrees in which objects (fully) in* A *have attribute* y *are all in the upper cone of* $A^{\uparrow}(y)$ *in* $\mathbf{L}$ *(Fig. 5 (left)). In the case* $A^{\uparrow}(y) = 0$*, objects (fully) in* A *may have the attribute* y *in any degree (Fig. 5 (middle)). In the case* $A^{\uparrow}(y) = 1$*, objects (fully) in* A *have the attribute* y *in full degree (Fig. 5 (right)). As positive information (having an attribute) is absolute in this setting, we say that the pair of concept-forming operators* $\langle {}^{\downarrow}, {}^{\uparrow} \rangle$ *considers attributes in a* positive *way – as affirmations. On the contrary, the truth degrees in which objects (fully) in* A *have attribute* y *are all in the lower cone of* $A^{\cap}(y)$ *in* $\mathbf{L}$ *(Fig. 6 (left)). In the case* $A^{\cap}(y) = 0$*, objects (fully) in* A *do not have the attribute* y*; i.e. they have it in degree 0. (Fig. 6 (middle)). In the case* $A^{\cap}(y) = 1$*, objects (fully) in* A *may have the attribute* y *in any degree (Fig. 6 (right)). As negative information (not having an attribute) is absolute in this setting, we say that the pair of concept-forming operators* $\langle {}^{\cup}, {}^{\cap} \rangle$ *considers attributes in a* negative *way – as denials.*

2.7 L-Concept Lattices

The pairs $\langle A, B \rangle \in L^X \times L^Y$, such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$, are called standard $\mathbf{L}$-concepts. Analogously, the pairs $\langle A, B \rangle \in L^X \times L^Y$, such that $A^{\cap} = B$ and $B^{\cup} = A$, are called attribute-oriented $\mathbf{L}$-concepts. The components A and B in standard or attribute-oriented $\mathbf{L}$-concept $\langle A, B \rangle$ are called extent and intent respectively.

The set of all formal concepts (along with set inclusion) forms a complete lattice, called $\mathbf{L}$-concept lattice. We denote the sets of all concepts (as well

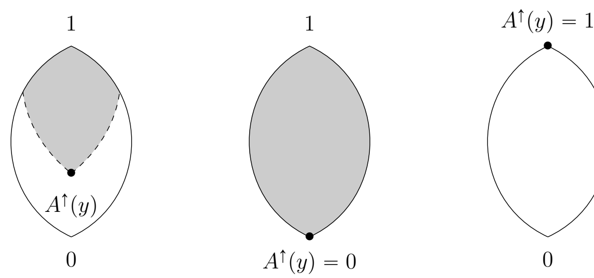


Figure 5: The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\uparrow}(y) = 0$ and $A^{\uparrow}(y) = 1$ (middle and right, respecively).

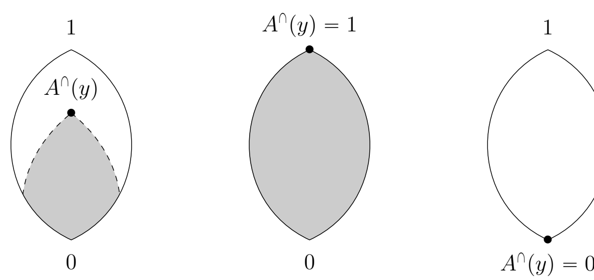


Figure 6: The truth degrees in which objects (fully) in A may have attribute y (gray area); general case (left), extreme cases $A^{\cap}(y) = 0$ and $A^{\cap}(y) = 1$ (middle and right, respecively).

as the corresponding **L**-concept lattice) by $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ and $\mathcal{B}^{\cap\cup}(X, Y, I)$, i.e.

$$\begin{aligned}\mathcal{B}^{\uparrow\downarrow}(X, Y, I) &= \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},\\ \mathcal{B}^{\cap\cup}(X, Y, I) &= \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\cap} = B,\ B^{\cup} = A\}.\end{aligned}$$

For an **L**-concept lattice $\mathcal{B}^{\#}(X, Y, I)$, where $\mathcal{B}^{\#}$ is either $\mathcal{B}^{\uparrow\downarrow}$ or $\mathcal{B}^{\cap\cup}$, denote the corresponding sets of extents and intents by $\mathrm{Ext}^{\#}(X, Y, I)$ and $\mathrm{Int}^{\#}(X, Y, I)$. That is,

$$\begin{aligned}\mathrm{Ext}^{\#}(X, Y, I) &= \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}^{\#}(X, Y, I) \text{ for some } B\},\\ \mathrm{Int}^{\#}(X, Y, I) &= \{B \in L^Y \mid \langle A, B\rangle \in \mathcal{B}^{\#}(X, Y, I) \text{ for some } A\}.\end{aligned}$$

See examples of standard and attribute oriented **L**-concept lattices depicted in Fig. 7 and Fig. 8).

2.8 Parameterization with Truth-Stressing Hedges

The standard concept-forming operators parameterized with the truth-stressing hedges were studied in [7, 12, 18][1]. The parametrization goes as follows: let $\langle X, Y, I\rangle$ be an **L**-context and let $*, \bullet$ be truth-stressing hedges on **L**. The standard concept-forming operators parameterized by $*$ and $\bullet$ induced by I are defined as

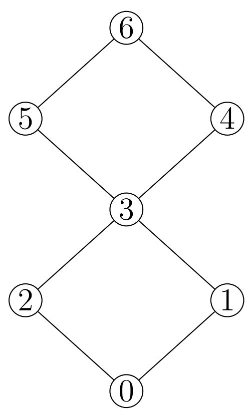
$$\begin{aligned}A^{\Uparrow}(y) &= \bigwedge_{x\in X}(A(x)^{*} \to I(x, y))\\ B^{\Downarrow}(x) &= \bigwedge_{y\in Y}(B(y)^{\bullet} \to I(x, y))\end{aligned} \tag{32}$$

for all $A \in L^X, B \in L^Y$.

The two boundary instances of hedges, namely $*$ being identity and globalization, are particularly important: With both truth-stressing hedges being identity, one obtains the standard fuzzy concept lattices of [9, 54], while for one of the truth-stressing hedge being globalization and the other being identity, one obtains the one-sided, or crisply generated, fuzzy concept lattices [19, 67, 47].

The meaning of $A^{\Uparrow}$ and $B^{\Downarrow}$ is essentially the same as that of their unhedged version. The difference is in that parts of the verbal description of

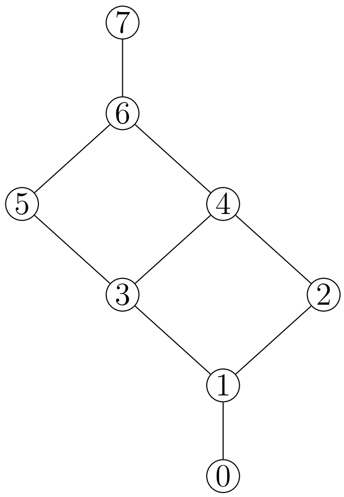
[1]Parameterization of attribute-oriented concept-forming operators is one of the contribution of this thesis, see Section A.



	extent	intent
⓪	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, \beta, \gamma\}$
①	$\{{}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$
②	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \beta, \gamma\}$
③	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
④	$\{{}^{0.5}/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$

$\preceq$	⓪	①	②	③	④	⑤	⑥
⓪	1	1	1	1	1	1	1
①	0.5	1	0.5	1	1	1	1
②	0.5	0.5	1	1	1	1	1
③	0.5	0.5	0.5	1	1	1	1
④	0	0.5	0	0.5	1	0.5	1
⑤	0	0	0.5	0.5	0.5	1	1
⑥	0	0	0	0.5	0.5	0.5	1

Figure 7: **L**-concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, I)$ (top left) of the **L**-context in Fig. 4, description of its **L**-concepts and the **L**-order $\preceq$ (bottom) .



	extent	intent
0	$\{{}^{0.5}/\mathrm{C}\}$	$\varnothing$
1	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\gamma\}$
2	$\{{}^{0.5}/\mathrm{A}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\beta, {}^{0.5}/\gamma\}$
3	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\gamma\}$
4	$\{{}^{0.5}/\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, {}^{0.5}/\gamma\}$
5	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{{}^{0.5}/\alpha, \gamma\}$
6	$\{\mathrm{A}, {}^{0.5}/\mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{{}^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
7	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$

$\preceq$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	0.5	1	0.5	1	1	1	1	1
2	0.5	0.5	1	0.5	1	0.5	1	1
3	0.5	0.5	0.5	1	1	1	1	1
4	0.5	0.5	0.5	0.5	1	0.5	1	1
5	0	0.5	0.5	0.5	0.5	1	1	1
6	0	0.5	0.5	0.5	0.5	0.5	1	1
7	0	0	0	0.5	0.5	0.5	0.5	1

Figure 8: **L**-concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ (top left) of the **L**-context in Fig. 4, description of its **L**-concepts and the **L**-order $\preceq$ (bottom) .

hedged version contains "very true" and "slightly true" respectively, compared to that of $A^{\uparrow}$ and $B^{\downarrow}$. For example, $A^{\uparrow}(y)$ is the truth degree of "all x for which it is *very true* that it belongs to A have attribute y".

Standard **L**-concepts with hedges $*, \bullet$ are pairs $\langle A, B\rangle \in L^X \times L^Y$ which satisfy $A^{\Uparrow} = B$ and $B^{\Downarrow} = A$. The set of all such concepts is denoted $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X, Y, I)$. The following theorem is an analogy to the main theorem on concept lattices.

Theorem 6. *1. $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X, Y, I)$ equipped with $\leqslant$, defined by*

$$\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2,$$

is a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{\bullet})^{\downarrow\uparrow}\rangle,$$
$$\bigvee_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcap_{j\in J} A_j^{*})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle.$$

2. *Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leqslant\rangle$ is isomorphic to $\mathcal{B}^{\uparrow\downarrow}{}_{*,\bullet}(X, Y, I)$ iff there are mappings $\mu : \mathrm{fix}(^*) \times X \to K$, $\nu : \mathrm{fix}(^{\square}) \times Y \to K$ such that*

   (a) $\mu(\mathrm{fix}(^*) \times X)$ is $\bigvee$-dense in K, $\nu(\mathrm{fix}(^{\bullet}) \times Y)$ is $\bigwedge$-dense in K.

   (b) $\mu(a, x) \leqslant \nu(b, y)$ iff $a \otimes b \leqslant I(x, y)$.

The reason for this parameterization is to have a tool to influence size of the number of concept lattice.

2.9 L-Attribute Implications

Attribute implications in the fuzzy setting with semantics corresponding to standard concept-forming operators were thoroughly studied in [20, 21].

Each expression of the form $A \Rightarrow B$, in which A and B are **L**-sets of attributes (i.e. $A, B \in L^Y$) is called a *fuzzy attribute implication* (FAI) over Y. Their intended meaning the same as in the ordinary case, that is:

if an object has all attributes in A it has also all attributes in B.

Since in a fuzzy setting, object-attribute incidence is a matter of degree, validity of our formulas is a matter of degree as well.

Let x denote an object and $M \in L^Y$ an **L**-set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. For the notion of validity, Belohlavek and Vychodil [20, 21] provide a general definition which subsumes two particular cases, one for bivalent and one for graded inclusion. For the bivalent inclusion, the fact that $A \Rightarrow B$ is fully true in M (in symbols $\|A \Rightarrow B\|_M = 1$) means:

$$\text{if } A \subseteq M \text{ then } B \subseteq M. \tag{33}$$

For the graded inclusion, the fact that $A \Rightarrow B$ is fully true in M means:

$$S(A, M) \leqslant S(B, M), \tag{34}$$

i.e. a degree of inclusion of A in M is less than or equal to the degree of inclusion of B in A, cf. (5). Both the approaches can be obtained as particular cases of the following definition.

$$\|A \Rightarrow B\|_M = S(A, M)^* \to S(B, M). \tag{35}$$

where the truth-stressing hedge * is used as a parameter.

For a collection $\mathcal{M}$ of fuzzy sets M of attributes in Y, we define the degree to which $A \Rightarrow B$ is valid in $\mathcal{M}$ as follows:

$$\|A \Rightarrow B\|_{\mathcal{M}} = \textstyle\bigwedge_{M \in \mathcal{M}} \|A \Rightarrow B\|_M. \tag{36}$$

For an **L**-context $\langle X, Y, I\rangle$, we define the degree to which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ by

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\{I_x \mid x \in X\}}, \tag{37}$$

where I_x denotes an **L**-set representing the row corresponding to object x, i.e. $I_x(y) = I(x, y)$ for each $y \in Y$.

For a fuzzy attribute implication $A \Rightarrow B$ and a fuzzy set M of attributes (of some object x) we define the *degree* $\|A \Rightarrow B\|_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as follows:

$$\|A \Rightarrow B\|_M = S(A, M)^* \to S(B, M). \tag{38}$$

One easily verifies that if * is globalization and identity, respectively, (42) meets the above cases corresponding to bivalent and graded inclusion, (40) and (41), respectively.

Given an $\mathbf{L}$-context $\langle X, Y, I\rangle$ and a FAI $A \Rightarrow B$ over Y, we have

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\mathrm{Int}^{\uparrow\downarrow}{}_{*}(X,Y,I)} \quad = S(B, A^{\downarrow\uparrow}). \tag{39}$$

A *theory* (over Y) is any set T of FAIs (over Y). The set $\mathrm{Mod}(T)$ of all *models* of a given theory T is then defined as

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A, B \in \mathbf{L}^Y : \\ T(A \Rightarrow B) \leqslant \|A \Rightarrow B\|_M\}.$$

$\mathrm{Mod}(T)$ is an $\mathbf{L}$-closure system.

We say that an FAI $A \Rightarrow B$ *semantically follows* from theory T, written $T \Vdash A \Rightarrow B$, if $A \Rightarrow B$ is valid in every model of T.

Bases We say that a theory is called

- *complete* in $\langle X, Y, I\rangle$ if for any FAI $A \Rightarrow B$ we have

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = 1 \text{ iff } T \Vdash A \Rightarrow B;$$

- *non-redundant* if for any $A \Rightarrow B \in T$ we have $T - \{A \Rightarrow B\} \Vdash A \Rightarrow B$;
- *basis* of $\langle X, Y, I\rangle$ if it is complete in $\langle X, Y, I\rangle$ and non-redundant.

We call a system $\mathcal{P}$ of fuzzy sets in Y a *system of pseudo-intents* (w.r.t. $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$) of $\mathbf{L}$-context $\langle X, Y, I\rangle$ if for every $\mathbf{L}$-set $P \in L^Y$ the following holds: $P \in \mathcal{P}$ iff $P \neq P^{\downarrow\uparrow}$ and for each $Q \in \mathcal{P}$ with $Q \neq P$ we have $\|Q \Rightarrow Q^{\downarrow\uparrow}\|_P = 1$.

Theorem 7. *Let $\langle X, Y, I\rangle$ be a formal context and $\mathcal{P}$ be a system of pseudo-intents. Then the theory*

$$\{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$$

is a basis of $\langle X, Y, I\rangle$.

The basis defined in Theorem 7 is called the *Guigues-Duquenne basis* [33, 29]. The main features of the Guigues-Duquenne basis in the ordinary setting are that it is unique (as exactly one system of pseudo-intents exist in the context), computationally tractable, and it is optimal in terms of its size; i.e. no other basis is smaller in terms of the number of FAIs it contains. It keeps these properties in the graded setting when globalization is used as the truth-stressing hedge.

3 Contributions of the Thesis

A Isotone Fuzzy Galois Connections with Hedges

[43] Jan Konecny. Isotone fuzzy Galois connections with hedges. *Information Sciences*, 181(10):1804–1817, 2011.

While the standard concept-forming operators parameterized with the truth-stressing hedges have been extensively studied [7, 12, 18], the attribute-oriented concept-forming operators received attention only in our works [2, 43].

We study attribute-oriented concept-forming operators and concept lattices parameterized by truth-stressing and truth-depressing hedges.

For a formal **L**-context $\langle X, Y, I\rangle$ we define a pair $\langle {}^{\cap}, {}^{\cup}\rangle$ of mappings ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ by

$$A^{\cap}(y) = \bigvee_{x\in X}(A(x)^{*} \otimes I(x, y)),$$
$$B^{\cup}(x) = \bigwedge_{y\in Y}(I(x, y) \to B(y)^{\square}).$$

The meaning of $A^{\cap}$ and $B^{\cup}$ is essentially the same as that of their unhedged version. The difference is that parts of the verbal description of the hedged version contains "very true" and "slightly true" respectively, compared to that of $A^{\cap}$ and $B^{\cup}$. For example, $A^{\cap}(y)$ is the truth degree of "there exists x for which it is very true that it belongs to A and which has y".

We study formal concepts and concept lattices $\mathcal{B}^{\cap\cup}{}_{*,\square}(X, Y, I)$ formed by the operators with hedges and provide an analogy of the main theorem for them.

We show that hedges enable us to control the number of formal **L**-concepts in the associated **L**-concept lattice. The whole point of generalizing the attribute-oriented concept-forming operators $\langle {}^{\cap}, {}^{\cup}\rangle$ by using a truth-stressing and truth-depressing hedge is to gain control over the size of the resulting **L**-concept lattice. In the case of the unhedged attribute-oriented concept-forming operators, the number of formal **L**-concepts can be inconveniently big.

In our previous work [2], we have studied a version of attribute-oriented concept-forming operators parameterized with truth-stressing hedges, specifically the pair $\langle \Cap, \Cup \rangle$ given by

$$A^{\Cap}(y) = \bigvee_{x \in X} (A(x)^{*} \otimes I(x, y)),$$
$$B^{\Cup}(x) = \bigwedge_{y \in Y} (I(x, y) \to B(y)^{\bullet})$$

where $*$ and $\bullet$ are both truth-stressing hedges. However, we demonstrated that the pair $\langle \cap, \cup \rangle$ provided better reduction of size than $\langle \Cap, \Cup \rangle$ as the reduction with the latter was too drastic and often led to a trivial two-element concept lattice.

Additionally, we provide a reduction theorem which enables us to elevate particular results valid in the ordinary setting into the graded setting with hedges.

B A Calculus for Containment of Fuzzy Attributes

[16] Radim Belohlavek and Jan Konecny. A calculus for containment of fuzzy attributes. *Soft Computing*, pages 1–12, 2017.

Dependencies in data describing objects and their attributes represent a key topic in understanding relational data. In this paper, we examine certain dependencies of data described by fuzzy attributes.

Each expression of the form

$$A \Rightarrow B,$$

in which A and B are fuzzy sets of attributes (i.e. $A, B \in L^Y$) is called a *fuzzy attribute implication* (FAI) over Y. While FAIs are identical with the formulas in Section 2.9 as far as syntax is concerned, their semantics is different. While those in Section 2.9 are linked to graded affirmations the present ones are linked to graded denials. Their intended meaning is:

> "if all attributes of an object are contained in A then they are contained in B"

or, in terms of the graded denials,

> "if an object has attributes at most to degrees given by A then it has attributes at most to degrees given by B."

Similarly, as in Section 2.9 two natural options for the formalization of the semantics are possible—assuming the containment as bivalent or as graded. We provide general semantics which covers both these options as particular cases.

Let x denote an object and $M \in L^Y$ an **L**-set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. We define the truth degree, denoted $\|A \Rightarrow B\|_M$, of $A \Rightarrow B$ in M, i.e. the truth degree to which $A \Rightarrow B$ is true for object x. For the bivalent containment, the fact that $A \Rightarrow B$ is fully true in M (in symbols $\|A \Rightarrow B\|_M = 1$) means

$$\text{if } M \subseteq A \text{ then } M \subseteq B. \tag{40}$$

For the graded containment, the fact that $A \Rightarrow B$ is fully true in M means

$$S(M, A) \leqslant S(M, B), \tag{41}$$

i.e. a degree of inclusion of M in A is less than or equal to the degree of inclusion of M in B, cf. (5). Analogously, as in Section 2.9, both the options can be obtained as particular cases of the following definition, in which the truth-stressing hedge * acts as a parameter. We define the *degree* $\|A \Rightarrow B\|_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as

$$\|A \Rightarrow B\|_M = S(M, A)^* \rightarrow S(M, B). \tag{42}$$

Among the main results established in the paper are: results regarding validity of dependencies, their models, and entailment; connections to existing dependencies for fuzzy as well as Boolean attributes, connections to interior- and closure-like structures, definition and properties of semantic entailment including an efficient check of entailment, various model-theoretical properties, a logical calculus of the dependencies inspired by the well-known Armstrong rules with its ordinary-style as well as graded-style syntactico-semantical completeness, basic results on bases, i.e. minimal fully informative sets of dependencies that are true in a given data.

C Concept Lattices of Isotone vs. Antitone Galois Connections in Graded Setting: Mutual Reducibility Revisited

[15] Radim Belohlavek and Jan Konecny. Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited. *Information Sciences*, 199:133–137, 2012.

It is well known that in the basic setting the standard and attribute-oriented concept lattices of a formal context and its complement are isomorphic, via a natural isomorphism which maps the extents to themselves and intents to their complements. It is also known that in the graded setting, this and similar kinds of reductions fail to hold. We show that when the usual notion of a complement, based on a residuum w.r.t. 0, is replaced by a new one, based on residua w.r.t. arbitrary truth degrees, the above-mentioned reduction remains valid.

On the one hand, it is well known that with **L** satisfying the double negation law ($\neg\neg a = a$) the attribute-oriented case is easily reducible to the standard case, and *vice versa*, via a set complement. Specifically, the attribute-oriented concept lattice $\mathcal{B}^{\cap\cup}(X, Y, I)$ is isomorphic to the standard concept lattice $\mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$. The isomorphism $i\colon \mathcal{B}^{\cap\cup}(X, Y, I) \to \mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I)$, as well as its inverse $i^{-1}\colon \mathcal{B}^{\uparrow\downarrow}(X, Y, \neg I) \to \mathcal{B}^{\cap\cup}(X, Y, I)$, is given by

$$i, i^{-1}\colon \langle A, B\rangle \mapsto \langle A, \neg B\rangle. \tag{43}$$

Clearly, we also have

$$\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y, \neg I). \tag{44}$$

On the other hand, this is no longer the case in the graded setting as the double negation law does not hold generally. We propose a new notion of complement of an **L**-relation: **L**-complement w.r.t. $K \subseteq L$ of an **L**-relation $I \in L^{X\times Y}$ is **L**-relation $\neg_K I \in L^{X\times(Y\times K)}$ given by

$$\neg_K I(x, \langle y, a\rangle) = I(x, y) \to a \tag{45}$$

for all $x \in X, y \in Y, a \in K$.

Utilizing this notion of complement, we can state one-way reducibility of the standard case to the attribute-oriented case:

Theorem 8. *Let $\langle X, Y, I\rangle$ be an $\mathbf{L}$-context. Then $\mathcal{B}^{\cap\cup}(X, Y, I)$ is isomorphic to $\mathcal{B}^{\uparrow\downarrow}(X, Y \times (L\backslash\{0\}), \neg_{L\backslash\{0\}} I)$ with $i : \langle A, A^{\cap}\rangle \mapsto \langle A, A^{\uparrow}\rangle$ being the isomorphism from $\mathcal{B}^{\cap\cup}(X, Y, I)$ to $\mathcal{B}^{\uparrow\downarrow}(X, Y \times L, \neg_{L\backslash\{0\}} I)$. Particularly,*

$$\mathrm{Ext}^{\cap\cup}(X, Y, I) = \mathrm{Ext}^{\uparrow\downarrow}(X, Y \times L, \neg_{L\backslash\{0\}} I).$$

The result reveals a new, deeper root of the reduction: it is not the availability of the law of double negation, but rather the fact that negations are implicitly present in the construction of attribute-oriented concept lattices.

A converse statement to Theorem 8 does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, the set $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ is equal to $\mathrm{Ext}^{\cap\cup}(X, Z, \sim I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}^{\uparrow\downarrow}(X, Y, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators $\langle^{\cap},^{\cup}\rangle$. This was demonstrated in [14].

D L-concept Analysis with Positive and Negative Attributes

[3] Eduard Bartl and Jan Konecny. L-concept analysis with positive and negative attributes. *Information Sciences*, 360:96–111, 2016.

We describe an extension of FCA in the graded setting, allowing a user to choose which incidences are viewed as affirmations and which are viewed as denials. The two sets are then handled using a combination of the standard and attribute-oriented concept-forming operators. Specifically, we extend the notion of formal **L**-context to contain two **L**-relations, ${}^{+}I$ and ${}^{-}I$, between objects and attributes. The membership degrees in ${}^{+}I$ present graded affirmations while the membership degrees in ${}^{-}I$ present graded denials. It is natural to assume that ${}^{+}I \subseteq {}^{-}I$. The intervals $[{}^{+}I(x, y), {}^{-}I(x, y)]$ are then seen as sets of truth degrees in which object x can have attribute y. As intents, we use pairs $\langle {}^{+}B, {}^{-}B\rangle \in L^Y \times L^Y$, where the **L**-sets ${}^{+}B$, ${}^{-}B$ respectively represent affirmations and and denials about attributes.

The concept-forming operators $\vartriangle\colon L^X \to L^Y \times L^Y$ and $\triangledown\colon L^Y \times L^Y \to L^X$ are defined as

$$A^{\vartriangle} = \langle A^{\uparrow}, A^{\cap}\rangle \text{ and } \langle {}^{+}B, {}^{-}B\rangle^{\triangledown} = {}^{+}B^{\downarrow} \cap {}^{-}B^{\cup} \tag{46}$$

for each $A \in L^X, {}^{+}B, {}^{-}B \in L^Y$; where the pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ is induced by $\langle X, Y, {}^{+}I\rangle$ and the pair $\langle {}^{\cap}, {}^{\cup}\rangle$ is induced by $\langle X, Y, {}^{-}I\rangle$.

Both the two main outputs of FCA are presented. In the first part, an analogy of the main theorem of concept lattices and a relationship between the new concept lattice and the previously studied concept lattices is shown.

In the second part, we describe the second main output of FCA. We present a general logic of if-then rules $\mathbf{A} \Rightarrow \mathbf{B}$ ($\mathbf{A}, \mathbf{B} \in L^Y \times L^Y$), called **L**-containment implications, for graded attributes which can be read: if all attributes of an object are contained in **A** then they are contained in **B**. Specifically, for $\mathcal{M} \subseteq L^Y \times L^Y$, the degree $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}}$ in which $\mathbf{A} \Rightarrow \mathbf{B}$ is valid in $\mathcal{M}$ is defined as $\|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathcal{M}} = \bigwedge_{\mathbf{M} \in \mathcal{M}} \|\mathbf{A} \Rightarrow \mathbf{B}\|_{\mathbf{M}}$.

E Rough Fuzzy Concept Analysis

[4] Eduard Bartl and Jan Konecny. Rough fuzzy concept analysis. *Fundamenta Informaticae*, 156(2):141–168, 2017.

We provide a new approach to the fusion of FCA in the graded setting and Rough Set Theory (RST). As a starting point we consider affirmations to represent the lower approximation, while the denails the upper approximation of a given input. Using the combination of concept-forming operators (46), we transfer the roughness of the input to the roughness of corresponding formal fuzzy concepts in the sense that a formal fuzzy concept is considered as a collection of objects accompanied with two fuzzy sets of attributes—those which are shared by all the objects and those which at least one object has. In the paper we study the properties of such formal concepts and show their relationship with concepts formed by well-known isotone and antitone operators.

We also demonstrate use RST inspired reduction of size of concept lattices based on equivalences induced by attributes and show that this reduction is natural, i.e. it preserves extents.

F Complete Relations on Fuzzy Complete Lattices

[45] Jan Konecny and Michal Krupka. Complete relations on fuzzy complete lattices. *Fuzzy Sets and Systems*, 320:64–80, 2017.

We focus on complete fuzzy tolerances. A (crisp) tolerance on a set is a reflexive and symmetric binary relation. A block of a tolerance is a set whose elements are pairwise related. A maximal block is a block which is maximal w.r.t. set inclusion. The set of all maximal blocks of a tolerance is called the factor set. One of basic results on tolerances on complete lattices in the basic setting is that complete lattices can be factorized by complete tolerances [28, 65]. That is, an ordering on the set of all maximal blocks of a complete tolerance can be introduced in a natural way, such that the factor set, together with this ordering, is again a complete lattice.

We show that this result hold true for complete **L**-tolerances on completely lattice **L**-ordered sets. More precisely, we use the usual definition of fuzzy tolerance and corresponding factor set and introduce an **L**-order on the factor set of a completely lattice **L**-ordered set by a complete **L**-tolerance, such that the new **L**-order is again a complete lattice **L**-order. To prove this main result, we more deeply investigate properties of complete **L**-tolerances. We use similar techniques to those used in classical ordered sets. However, we also introduce a result that is new even in the classical case: we show that complete fuzzy tolerances are in one-to-one correspondence with so-called extensive isotone fuzzy Galois connections.

G Block Relations in Formal Fuzzy Concept Analysis

[44] Jan Konecny and Michal Krupka. Block relations in formal fuzzy concept analysis. *International Journal of Approximate Reasoning*, 73:27–55, 2016.

One of the main problems in FCA, especially in the graded setting, is to reduce a concept lattice of a formal context to an appropriate size to make it graspable and understandable by a human user. A natural way to do it is to substitute the formal context by its block relation which is equivalent to factorization of the concept lattice by a complete tolerance. We generalize the known results on the correspondence of block relations of formal contexts and complete tolerances on concept lattices to the graded setting.

We provide a definition of block **L**-relation—a convenient generalization of the notion of block-relation from [66]. We show, that the block **L**-relations are in one-to-one correspondence to particular automorphisms on concept lattices. We describe the structure of systems of all block **L**-relations. All the results are considered for all three kinds of concept-forming operators.

H On Homogeneous L-bonds and Heterogeneous L-bonds

[46] Jan Konecny and Manuel Ojeda-Aciego. On homogeneous L-bonds and heterogeneous L-bonds. *International Journal of General Systems*, 45(2):160–186, 2016.

In this paper, we deal with suitable generalizations of the notion of bond between contexts. We study different generalizations of the notion of bond within the L-fuzzy setting. Specifically, given a formal context, there are three prototypical pairs of concept-forming operators, and this immediately leads to three possible versions of the notion of bond (so-called homogeneous bond w.r.t. a certain pair of concept-forming operators). The first results show a close correspondence between a homogeneous bond between two contexts and certain special types of mappings between the sets of extents (or intents) of the corresponding concept lattices. Later, we introduce the so-called heterogeneous bonds (considering simultaneously two types of concept-forming operators) and generalize the previous relationship to mappings between the sets of extents (or intents) of the corresponding concept lattices.

For all the defined bonds we provide their characterization, description of a structure they form and their relationship to direct products of relations. Finally, we explain the relationship of the bonds to the morphisms of **L**-closure systems and **L**-interior systems.

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