Structuring Digital Spaces by Path-Partition Induced Closure Operators on Graphs

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Abstract. We study closure operators on graphs which are induced by path partitions, i.e., certain sets of paths of the same lengths in these graphs. We investigate connectedness with respect to the closure operators studied. In particular, the closure operators are discussed that are induced by path partitions of some natural graphs on the digital spaces $\mathbb{Z}^n$, $n > 0$ a natural number. For the case $n = 2$, i.e., for the digital plane $\mathbb{Z}^2$, the induced closure operators are shown to satisfy an analogue of the Jordan curve theorem, which allows using them as convenient background structures for studying digital images.

Keywords: Graphs · Closure operators · Path-partition · Connectedness · Jordan curve

1 Introduction

The classical approach to digital topology is based on graph theory rather than topology because it uses adjacency graphs for structuring the digital spaces (see, e.g., [6]). In the case of the digital plane $\mathbb{Z}^2$, the most usual of these graphs are the 4-adjacency and 8-adjacency graphs. Unfortunately, neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem - cf. [9]. Therefore, one has to use a combination of the two adjacency graphs when studying (the connectivity of) digital images. To eliminate this deficiency, a new, purely topological approach to digital topology was proposed in [5]. This approach utilizes a convenient topology on $\mathbb{Z}^2$, the so-called Khalimsky topology, for structuring the digital plane and has been developed by many authors. For instance, some other convenient topologies on $\mathbb{Z}^2$ were introduced and studied in [14] and, in [13], it was shown that even closure operators, i.e., structures more general than topologies, may be used to advantage for structuring the digital spaces.

Each of the two approaches, classical and topological, has its advantages and it may, therefore, be beneficial to use a combination of them. Being motivated by this idea, we deal with closure operators on graphs. More precisely, we discuss closure operators on (the vertex sets of) graphs that are induced by path partitions of these graphs. The closure operators are studied from the viewpoint

R.P. Barneva et al. (Eds.): CompIMAGE 2016, LNCS 10149, pp. 43–55, 2017.
DOI: 10.1007/978-3-319-54609-4_3

of applications of the results obtained in digital topology for structuring the digital spaces. We will, therefore, focus on the connectedness provided by the closure operators. It will be shown that products of the operators preserve the basic properties of connectedness. Thus, having a path-partition induced closure operator on the digital line $\mathbb{Z}$, the connectedness behavior of the closure operator is also preserved by the corresponding product closure operator on the digital space $\mathbb{Z}^n$, $n > 1$ a natural number. We will discuss special path-partition induced closure operators on $\mathbb{Z}$ which generalize the Khalimsky topology. The products of pairs of copies of these closure operators will be shown to allow for a digital analogue of the Jordan curve theorem, i.e., to provide convenient structures on the digital plane $\mathbb{Z}^2$ for the study of digital images.

Graphs with path partitions were introduced and studied in [15] where it was shown that path partitions provide graphs with special geometric properties that allow for using these graphs in digital topology for the study of digital images. In the present paper, we investigate the topological properties of these graphs via induced closure operators. The results obtained pave the way for further applications of the graphs with path partitions in digital topology.

2 Preliminaries

For the graph-theoretic terminology, we refer to [4]. By a *graph* $G = (V, E)$ we understand an (undirected simple) graph (without loops) with $V \neq \emptyset$ the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ the set of *edges*. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$, and an edge $e \in E$ and a vertex $x \in V$ are said to be *adjacent* if $x \in e$. Throughout the paper, natural numbers are considered to be finite ordinals (including 0). Thus, given a natural number n, we write $(x_i|\ i \leq n)$ to denote the (finite) sequence $(x_0, x_1, ..., x_n)$. Similarly, if $n > 0$, we write $(x_i|\ i < n)$ to denote the sequence $(x_0, x_1, ..., x_{n-1})$.

Recall that a *walk* in G is a sequence $(x_i|\ i \leq n)$ of vertices of V such that x_i is adjacent to x_{i+1} whenever $i < n$. The natural number n is called the *length* of the walk $(x_i|\ i \leq n)$. If $C_1 = (x_i|\ i \leq m)$ and $C_2 = (y_i|\ i \leq q)$ are walks in G such that $x_m = y_0$, then we define $C_1 \oplus C_2 = (z_i|\ i \leq m + q)$ where $z_i = x_i$ for all $i \leq m$ and $z_i = y_{i-m}$ for all i with $m < i \leq m + q$. The walk $C_1 \oplus C_2$ (of length $n + m$) is called the *composition* of C_1 and C_2.

A walk $(x_i|\ i \leq n)$ is said to be *closed* if $x_0 = x_n$, and it is said to be a *path* if $x_i \neq x_j$ whenever $i, j \leq n$, $i \neq j$. A closed walk $(x_i|\ i \leq n)$ is called a *circle* if $(x_i|\ i < n)$ is a path. We will often apply set-theoretic operations on walks, in which case the walks will be considered to be just sets (a walk $(x_i|\ i \leq n)$ will be considered to be the set $\{x_i;\ i \leq n\}$). Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2.

Given graphs $G_j = (V_j, E_j,)$, $j = 1, 2, ..., m$ ($m > 0$ a natural number), we define their *product* to be the graph $\prod_{j=1}^m G_j = (\prod_{j=1}^m V_j, E)$ with the set of edges $E = \{\{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\}$; there exists a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $\{x_j, y_j\} \in E_j$ for every $j \in J$ and $x_j = y_j$

for every $j \in \{1, 2, ..., m\} - J\}$. This product differs from the cartesian product of G_j, $j = 1, 2, ..., m$, i.e., from the graph $(\prod_{j=1}^{m} V_j, F)$ where $F = \{\{(x_1, x_2, ..., x_3), (y_1, y_2, ..., y_m)\};\ \{x_j, y_j\} \in E_j$ for every $j \in \{1, 2, ..., m\}\}$, but we always have $F \subseteq E$.

By a *closure operator* u on a set X we mean a map u: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is

(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).

The pair (X, u) is then called a *closure space* and, for every subset $A \subseteq X$, uA is called the *closure* of A. Closure spaces were studied by Čech in [1] (who called them topological spaces there).

A closure operator u on X which is

(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$)

is called a *Kuratowski closure operator* or a *topology* and the pair (X, u) is called a *topological space*.

A closure operator u on a set X is said to be *quasi-discrete* (cf. [2]) if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup_{x \in A} u\{x\}$.

Thus, any quasi-discrete closure operator is additive and is given by closures of the singleton subsets. The quasi-discrete closure operators that are idempotent are called *Alexandroff topologies*.

Many concepts known for topological spaces (see e.g. [3]) may be naturally extended to closure spaces. Given a closure space (X, u), a subset $A \subseteq X$ is called *closed* if $uA = A$, and it is called *open* if $X - A$ is closed. A closure space (X, u) is said to be a *subspace* of a closure space (Y, v) (or, briefly, X is a subspace of (Y, v)) if $uA = vA \cap X$ for each subset $A \subseteq X$. A closure space (X, u) is said to be *connected* if $\emptyset$ and X are the only subsets of X which are both closed and open. A subset $X \subseteq Y$ is connected in a closure space (Y, v) if the subspace X of (Y, v) is connected. A maximal connected subset of a closure space is called a *component* of this space. All the basic properties of connected sets and components in topological spaces are also preserved in closure spaces.

3 Graphs with n-partitions and the Associated Closure Operators

From now on, we suppose that there is given a natural number $n > 0$. For any graph G, we denote by $\mathcal{P}_n(G)$ the set of all paths of length n in G. Observe that $\mathcal{P}_n(G)$ is nothing but an n-ary relation on the vertex set of G and so are the subsets of $\mathcal{P}_n(G)$ (cf. [11]).

Definition 1. Let G be a graph. A subset $\mathcal{B} \subseteq \mathcal{P}_n(G)$ is said to be an *n-path partition*, briefly an *n-partition*, of G provided that

(i) for every edge e of G, there is a unique path $(x_i|\ i \leq n) \in \mathcal{B}$ with the property that there exists $i \in \{1, 2, ...n\}$ such that e is adjacent to x_{i-1} and x_i,
(ii) every pair of different paths belonging to $\mathcal{B}$ has at most one vertex in common.

Clearly, every graph has a 1-partition. But, for $n > 1$, a graph need not have any n-partition.

Let G be a graph and $\mathcal{B}$ an n-partition of G. Then we define

$\mathcal{B}^{-1} = \{(x_i|\ i \leq n) \in \mathcal{P}_n(G);\ (x_{n-i}|\ i \leq n) \in \mathcal{B}\}$,
$\hat{\mathcal{B}} = \{(x_i|\ i \leq m) \in \mathcal{P}_m(G);\ 0 < m \leq n$ and there exists $(y_i|\ i \leq n) \in \mathcal{B}$ such that $x_i = y_i$ for every $i \leq m\}$ (so that $\mathcal{B} \subseteq \hat{\mathcal{B}}$), and
$\mathcal{B}^* = \hat{\mathcal{B}} \cup \hat{\mathcal{B}}^{-1}$.

Thus, $\mathcal{B}^{-1}$ is obtained by just reversing the sequences belonging to $\mathcal{B}$ and $\hat{\mathcal{B}}$ consists of just the sequences that are initial parts of the sequences belonging to $\mathcal{B}$. The elements of $\mathcal{B}^*$ will be called $\mathcal{B}$-*initial segments* in G - they are just the initial parts of the sequences belonging to $\mathcal{B}$ and the sequences that are their reversals.

Let G be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. For any subset $X \subseteq V$, we define $u_{\mathcal{B}}X = X \cup \{x \in V$; there exists $(x_i|\ i \leq m) \in \hat{\mathcal{B}}$ with $\{x_i;\ i < m\} \subseteq X$ and $x_m = x\}$.

Evidently, $u_{\mathcal{B}}$ is a closure operator on V (which is neither additive nor idempotent in general). It will be said to be *induced* by $\mathcal{B}$. It may easily be shown that $(V, u_{\mathcal{B}})$ is an Alexandroff topological space whenever $u_{\mathcal{B}}$ is idempotent.

Definition 2 ([15]). Let G be a graph with an n-partition $\mathcal{B}$. A finite nonempty sequence $C = (x_i|\ i \leq m)$ ($m > 0$ a natural number) of vertices of G is called a $\mathcal{B}$-*walk* in G if there is a finite increasing sequence $(j_k|\ k \leq p)$ of natural numbers with $j_0 = 0$ and $j_p = m$ such that $j_k - j_{k-1} \leq n$ and $(x_j|\ j_{k-1} \leq j \leq j_k) \in \mathcal{B}^*$ for every k with $k \leq p$. The sequence $(j_k|\ k \leq p)$ is said to be a *binding sequence* of C. If $x_0 = x_m$, then the $\mathcal{B}$-path C is said to be *closed* and, if for any pair i_0, i_1 of natural numbers with $i_0 < i_1 \leq m$ from $x_{i_0} = x_{i_1}$ it follows that $i_0 = 0$ and $i_1 = m$, then C is called a $\mathcal{B}$-*circle*. A $\mathcal{B}$-walk $(x_i|\ i < m)$ in G is said to *connect* points $x, y \in X$ if $\{x, y\} = \{x_0, x_m\}$.

Thus, also one-member sequences are considered to be $\mathcal{B}$-walks (and $\mathcal{B}$-circles) - their binding sequences have just one member, the number 0. It is evident that the composition of $\mathcal{B}$-walks is a $\mathcal{B}$-walk. If $\mathcal{B} \subseteq \mathcal{P}_1(G)$, then every $\mathcal{B}$-path in G is a path in G, and vice versa provided that $\mathcal{B} = \mathcal{P}_1(G)$. By [15], Theorem 1, every $\mathcal{B}$-walk in a graph G has a unique binding sequence.

Theorem 1. *Let $G = (V, E)$ be a graph with an n-partition $\mathcal{B}$. A subset $A \subseteq V$ is connected in the closure space $(V, u_{\mathcal{B}})$ if and only if any two different vertices of G belonging to A can be joined by a $\mathcal{B}$-walk in G contained in A.*

Proof. If A is empty or a singleton, then the statement is trivial. Therefore, suppose that A has at least two elements. Let any two vertices from A can be joined by a $\mathcal{B}$-walk in G contained in A. Then, choosing an arbitrary element $x \in A$, A is the union of the $\mathcal{B}$-walks in G joining x with the elements of $A-\{x\}$. Therefore, A is connected in $(V, u_{\mathcal{B}})$.

Conversely, let A be connected in $(V, u_{\mathcal{B}})$ and suppose that there are vertices $x, y \in A$ which cannot be joined by a $\mathcal{B}$-walk in G contained in A. Let B be the set of all vertices in A which can be joined with x by a $\mathcal{B}$-walk in G contained in A. Let $z \in u_{\mathcal{B}}B \cap A$ be a vertex and assume that $z \notin B$. Then there are a path $(x_i|\ i \le n) \in \mathcal{B}$ and a natural number i_0, $0 < i_0 \le n$, such that $z = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq B$. Thus, x and x_0 can be joined by a $\mathcal{B}$-walk in G contained in A, and also x_0 and z can be joined by a $\mathcal{B}$-walk in G contained in A - namely by the $\mathcal{B}$-initial segment $(x_i|\ i \le i_0) \in \mathcal{B}^*$. It follows that x and z can be joined by a $\mathcal{B}$-walk in G contained in A, which is a contradiction. Therefore, $z \in B$, which yields $u_{\mathcal{B}}B \cap A \subseteq B$. As the converse inclusion is evident, we have $u_{\mathcal{B}}B \cap A = B$. Consequently, B is closed in the subspace A of $(V, u_{\mathcal{B}})$. Further, let $z \in u_{\mathcal{B}}(A-B) \cap A$ be a vertex and assume that $z \in B$. Then $z \notin A - B$, thus there are a walk $(x_i|\ i < n) \in \mathcal{B}$ and a natural number i_0, $0 < i_0 \le n$, such that $z = x_{i_0}$ and $\{x_i;\ i < i_0\} \subseteq A - B$. Since x can be joined with z by a $\mathcal{B}$-walk in G contained in A (because we have assumed that $z \in B$) and z can be joined with x_0 by a $\mathcal{B}$-walk in G contained in A - namely by the initial segment $(x_{i_0-i}|\ i \le i_0) \in \mathcal{B}^*$, also x and x_0 can be joined by a $\mathcal{B}$-walk in G contained in A. This is a contradiction with $x_0 \notin B$. Thus, $z \notin B$, i.e., $u_{\mathcal{B}}(A-B) \cap A = A-B$. Consequently, $A-B$ is closed in the subspace A of $(V, u_{\mathcal{B}})$. Hence, A is the union of the nonempty disjoint sets B and $A - B$ closed in the subspace A of $(V, u_{\mathcal{B}})$. But this is a contradiction because A is connected in $(V, u_{\mathcal{B}})$. Therefore, any two points of A can be joined by a $\mathcal{B}$-walk in G contained in A. $\square$

Clearly, every $\mathcal{B}$-segment (and thus also every $\mathcal{B}$-path) in G is a connected subset of $(V, u_{\mathcal{B}})$.

Proposition 1. *Let G_j be a graph, $\mathcal{B}_j$ an n-partition of G_j for every $j = 1, 2, ..., m$ ($m > 0$ a natural number). Then $\prod_{j=1}^{m} \mathcal{B}_j$ is an n-partition of the graph $\prod_{j=1}^{m} G_j$.*

Proof. Clearly, $\prod_{j=1}^{m} \mathcal{B}_j \subseteq \mathcal{P}_n(\prod_{j=1}^{m} G_j)$. Let $e = \{(x_1, x_2, ..., x_m), (y_1, y_2, ..., y_m)\}$ be an edge in the graph $\prod_{j=1}^{m} G_j$. Then there is a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $\{x_j, y_j\}$ is an edge in G_j for every $j \in J$ and $x_j = y_j$ for every $j \in \{1, 2, ..., m\} - J$. Thus, for every $j \in J$, there is a unique path $(y_i^j|\ i \le n)$ with the property that there is $i_j \in \{1, 2, ..., n\}$ such that $\{x_j, y_j\} = \{z_{i_j-1}^j, z_{i_j}^j\}$. For every $i \le n$, put $t_i^j = z_i^j$ whenever $j \in J$ and $t_i^j = x_j$ whenever $j \in \{1, 2, ..., m\} - J$. Evidently, $((x_i^1, x_i^2, ..., x_i^m)|\ i \le n) \in \prod_{j=1}^{m} \mathcal{B}_j$ and it is a unique path with the property that there exists $i \in \{1, 2, ..., n\}$ such that $e = \{(x_{i-1}^1, x_{i-1}^2, ..., x_{i-1}^m), (x_i^1, x_i^2, ..., x_i^m)\}$.

Let $C, D \in \prod_{j=1}^{m} \mathcal{B}_j$ be different paths, $C = ((x_i^1, x_i^2, ..., x_i^m)|\ i \le n)$, $D = ((y_i^1, y_i^2, ..., y_i^m)|\ i \le n)$. Then there is $J_1 \subseteq \{1, 2, ..., m\}$ such that $(x_i^j|\ i \le n) \in \mathcal{B}_j$ for every $j \in J_1$ and $(x_i^j|\ i \le n)$ is a constant sequence for every $j \in \{1, 2, ..., m\} - J_1$. Similarly, there is $J_2 \subseteq \{1, 2, ..., m\}$ such that $(x_i^j|\ i \le$

$n) \in \mathcal{B}_j$ for every $j \in J_2$ and $(x_i^j|\ i \leq n)$ is a constant sequence for every $j \in \{1,2,...,m\} - J_2$. Since C and D are different, there are $j_0 \in \{1,2,...,m\}$ and a natural number $i_0 \leq n$ such that $x_{i_0}^{j_0} \neq y_{i_0}^{j_0}$. If $j_0 \in J_1 \cap J_2$, then $(x_i^{j_0}|\ i \leq n) \in \mathcal{B}_{j_0}$ and $(y_i^{j_0}|\ i \leq n) \in \mathcal{B}_{j_0}$ are different paths, so that they may have at most one vertex in common. Therefore, C and D have at most one vertex in common, too. If $j_0 \in J_1 - J_2$, then $(x_i^{j_0}|\ i \leq n) \in \mathcal{B}_{j_0}$ and $(y_i^{j_0}|\ i \leq n)$ is a constant sequence. Hence, C and D have at most one vertex in common. If $j_0 \in J_2 - J_1$, then the situation is analogous. Finally, suppose that $j_0 \notin J_1$ and $j_0 \notin J_2$. Then $(x_i^{j_0}|\ i \leq n)$ and $(y_i^{j_0}|\ i \leq n)$ are different constant sequences, so that C and D have no vertex in common. The proof is complete. □

Let G_j be a graph and $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$ for every $j = 1,2,...,m$ ($m > 0$ a natural number). The previous Proposition enables us to define $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j}) = (\prod_{j=1}^m V_j, u_{\prod_{j=1}^m \mathcal{B}_j})$.

Theorem 2. *Let $G_j = (V_j, E_j)$ be a graph, $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$, and $Y_j \subseteq V_j$ be a subset for every $j = 1,2,...,m$. If Y_j is a connected subset of $(V_j, u_{\mathcal{B}_j})$ for every $i = 1,2,...,n$, then $\prod_{j=1}^m Y_j$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$.*

Proof. Let Y_j be a connected subset of $(X_j, u_{\mathcal{B}_j})$ for every $j \in \{1,2,...,m\}$ and let $(y_1, y_2, ..., y_m), (z_1, z_2, ..., z_m) \in \prod_{j=1}^m Y_j$ be arbitrary points. By Theorem 1, for every $j \in \{1,2,...,m\}$, there is a $\mathcal{B}_j$-walk $(x_i^j|\ i \leq p_j)$ in G_j joining the points y_j and z_j which is contained in Y_j. Then the set $\prod_{j=1}^m \{x_i^j|\ i \leq p_j\}$ contains the points $(y_1, y_2, ..., y_m)$ and $(z_1, z_2, ..., z_m)$. For each $j = 1,2,...,m$, let $(i_k^j|\ k \leq q_j)$ be the binding sequence of $(x_i^j|\ i \leq p_j)$. For every $j = 1,2,...,m$, putting $C_k^j = \{x_i^j;\ i_k^j \leq i \leq i_{k+1}^j\}$ for all $k < q_j$, we get $\{x_i^j;\ i \leq p_j\} = \bigcup_{k<q_j} C_k^j$. Therefore, $\prod_{j=1}^m \{x_i^j;\ i \leq p_j\} = \bigcup_{k_1<q_1} \bigcup_{k_2<q_2} \cdots \bigcup_{k_m<q_m} \prod_{j=1}^m C_{k_j}^j$. Put $C_{k_j}^j = \{y_i^j;\ i \leq r_j\}$ for every $j \in \{1,2,...,m\}$ and every $k_j < q_j$. For each $j = 1,2,...,m$, there is a path $(z_i^j|\ i \leq n) \in \mathcal{B}$ such that $y_i^j = z_i^j$ for all $i \leq r_j$ or $y_i^j = z_{r_j-i}^j$ for all $i \leq r_j$ (because $(y_i^j|\ i \leq r_j)$ is a $\mathcal{B}_j$-initial segment in G_j). Let $y \in \prod_{j=1}^m \{y_i^j;\ i \leq r_j\}$ be an arbitrary element. Then, for each $j = 1,2,...,m$, there is a natural number s_j, $s_j < r_j$, such that $y = (y_{s_1}^1, y_{s_2}^2, ..., y_{s_m}^m)$. Then either $(y_{s_1-i}^1|\ i \leq s_1)$ or $(y_i^1|\ s_1 \leq i \leq r_1)$ is a $\mathcal{B}_1$-initial segment in G_1 with the first member $y_{s_1}^1$ and the last one x_0^1. Denote this $\mathcal{B}_1$-initial segment by $(t_i^1|\ i \leq u_1)$ and put $C_1 = ((t_i^1, y_{s_2}^2, y_{s_3}^3, ..., y_{s_m}^m)|\ i \leq u_1)$. Clearly, C_1 is a $\prod_{j=1}^m \mathcal{B}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq r_j\}$, with the first member y, and with $t_{u_1}^1 = x_0^1$. Further, either $(y_{s_2-i}^2|\ i \leq s_2)$ or $(y_i^2|\ s_2 \leq i \leq r_2)$ is a $\mathcal{B}_2$-initial segment in G_2 with the first member $y_{s_2}^2$ and the last one x_0^2. Denote this $\mathcal{B}_2$-initial segment by $(t_i^2|\ i \leq u_2)$ and put $C_2 = ((z_0^1, t_i^2, y_{s_3}^3, y_{s_4}^4, ..., y_{s_m}^m)|\ i \leq u_2)$. Clearly, C_2 is a $\prod_{j=1}^m \mathcal{B}_j$-initial segment in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq r_j\}$ such that $t_0^2 = y_{s_2}^2$ and $t_{u_2}^2 = x_0^2$. Thus, $C_1 \oplus C_2$ is a $\prod_{j=1}^m \mathcal{B}_j$-walk in $\prod_{j=1}^m G_j$ with all members belonging to $\prod_{j=1}^m \{y_i^j;\ i \leq r_j\}$, with the first member y, and with the last one

$(z_0^1, z_0^2, y_{s_3}^3, y_{s_4}^4, ..., y_{s_m}^m)$. Repeating this construction m-times, we get $\prod_{j=1}^m \mathcal{B}_j$-initial segments C_1, C_2,...,C_m in $\prod_{j=1}^m G_j$ with the members of each of them belonging to $\prod_{j=1}^m \{y_i^j;\ i \le r_j\}$ such that $C_1 \oplus C_2 \oplus ... \oplus C_m$ is a $\prod_{j=1}^m \mathcal{B}_j$-walk in $\prod_{j=1}^m G_j$ with the first member y and the last one $(z_0^1, z_0^2, ..., z_0^m)$. Then any point of $\prod_{j=1}^m \{y_i^j;\ i \le r_j\}$ can be joined with the point $(x_0^1, x_0^2, ..., x_0^m)$ by a $\prod_{j=1}^m \mathcal{B}_j$-walk in $\prod_{j=1}^m G_j$ contained in $\prod_{j=1}^m \{y_i^j;\ i \le r_j\}$. By Theorem 1, $\prod_{j=1}^m \{y_i^j;\ i \le r_j\} = \prod_{j=1}^m C_{k_j}^j$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$. Thus, for any $k_j < q_j$, $j = 1, 2, ..., m-1$, $(\prod_{j=1}^m C_{k_j}^j \mid k_m < q_m)$ is a finite sequence of connected sets with nonempty intersection of every consecutive pair of them. Hence, the set $\bigcup_{k_m < q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$. Consequently, for every k_j with $k_j < q_j$, $j = 1, 2, ..., m-2$, $(\bigcup_{k_m < q_m} \prod_{j=1}^m C_{k_j}^j \mid k_{m-1} < q_{m-1})$ is a finite sequence of connected sets with nonempty intersection of any consecutive pair of them. Therefore, the set $\bigcup_{k_{m-1} < q_{m-1}} \bigcup_{k_m < q_m} \prod_{j=1}^m C_{k_j}^j$ is connected in $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$. After repeating this considerations m-times, we get the conclusion that the set $\bigcup_{k_1 < q_1} \bigcup_{k_2 < q_2} \cdots \bigcup_{k_m < q_m} \prod_{j=1}^m C_{k_j}^j = \prod_{j=1}^m \{x_i^j;\ i \le p_j\}$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$. Thus, by Theorem 1, there is a $\mathcal{B}_j$-walk C in $\prod_{j=1}^m G_j$ joining the points $(y_1, y_2, ..., y_m)$ and $(z_1, z_2, ..., z_m)$ which is contained in $\prod_{j=1}^m \{x_i^j \mid i \le p_j\}$. Since $\prod_{j=1}^m \{x_i^j \mid i \le p_j\} \subseteq \prod_{j=1}^m Y_j$, C is contained in $\prod_{j=1}^m Y_j$, too, and so $\prod_{j=1}^m Y_j$ is a connected subset of $\prod_{j=1}^m (V_j, u_{\mathcal{B}_j})$ by Theorem 1. $\square$

4 Graphs on $\mathbb{Z}^n$ with an n-partition

In this section, we show that, for a number of structures on $\mathbb{Z}^2$ used in digital topology (including the Marcus-Wise and Khalimsky topologies), the connectedness they provide coincides with the connectedness with respect to a closure operator $u_{\mathcal{B}}$ for a suitable n-partition $\mathcal{B}$ of a graph G on $\mathbb{Z}^2$. After discussing some graphs with a 1-partition, we will focus on a certain type of graphs with an n-partition where $n > 1$ may be an arbitrary natural number.

It is well known that closure operators that are more general than the Kuratowski ones have useful applications in computer science. By Theorem 1, connectedness with respect to the closure operators on graphs induced by path partitions is a certain type of path connectedness, which enables us to apply graph-theoretic methods when studying these closure operators. This may especially be useful for applications of the closure operators in digital topology, a branch of digital geometry built for the study of geometric and topological properties of digital images (cf. [7,10]). For such applications, it is desirable that the closure operators satisfy a digital analogue of the Jordan curve theorem. The classical Jordan curve theorem states that every simple closed curve in the real (i.e., Euclidean) plane separates this plane into precisely two connected components.

Recall that the 8-*adjacency graph* and the 4-*adjacency graph* are the graphs $(\mathbb{Z}^2, A_4)$ and $(\mathbb{Z}^2, A_8)$ given by $A_4 = \{\{(x_1, y_1), (x_2, y_2)\};\ |x_1 - x_2| + |y_1 + y_2| = 1\}$ and $A_8 = \{\{(x_1, y_1), (x_2, y_2)\};\ (x_1, y_1), (x_2, y_2) \in \mathbb{Z}^2,\ \max\{|x_1 - x_2|, |y_1 - y_2|\} = 1\}$, respectively. The points adjacent (to each other) in $(\mathbb{Z}^2, A_4)$ are called 4-*adjacent* and those adjacent in $(\mathbb{Z}^2, A_8)$ are called 8-*adjacent*. Given a point $z \in \mathbb{Z}^2$, we denote by $A_4(z)$ the set of points that are 4-adjacent to z and by $A_8(z)$ the set of points that are 8-adjacent to z.

There are two well-known topologies on $\mathbb{Z}^2$ allowing for an analogue of the Jordan curve theorem, which are employed in digital topology. These are the so-called *Marcus-Wyse* and *Khalimsky* topologies, i.e., the Alexandroff topologies s and t, respectively, on $\mathbb{Z}^2$ with the closures of singleton subsets given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$s\{z\} = \begin{cases} \{z\} \cup A_4(z) \text{ if } x + y \text{ is even,} \\ \{z\} \text{ otherwise} \end{cases}$$

and

$$t\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x, y \text{ are even,} \\ \{(x+i, y); i \in \{-1, 0, 1\}\} \text{ if } x \text{ is even and } y \text{ is odd,} \\ \{(x, y+j); j \in \{-1, 0, 1\}\} \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{z\} \text{ otherwise.} \end{cases}$$

We will show that both of these topologies may be obtained as closure operators induced by an n-path set in certain graph with the vertex set $\mathbb{Z}^2$.

Let $G = (\mathbb{Z}^2, E)$ where $E = \{\{(x, y), (z, t)\};\ (x, y), (z, t) \in \mathbb{Z}^2, |x-z|+|y-t| = 1\}$ and define $\mathcal{B} = \{((x_i, y_i)|\ i \leq 1);\ (x_i, y_i) \in \mathbb{Z}^2 \text{ for every } i \leq 1,\ |x_0 - x_1| + |y_0 - y_1| = 1,\ x_0 + y_0 \text{ even}\}$. Then $\mathcal{B}$ is a 1-partition of G.

It may easily be seen that $(\mathbb{Z}^2, u_{\mathcal{B}})$ is a connected Alexandroff topological space in which the points $(x, y) \in \mathbb{Z}^2$ with $x + y$ even are open while those with $x + y$ odd are closed. The closure operator $u_{\mathcal{B}}$ coincides with the Marcus-Wyse topology.

We denote by $\mathbb{Z}_2$ the 2-*adjacency graph* on $\mathbb{Z}$, i.e., the graph $(\mathbb{Z}, A_2)$ where $A_2 = \{\{p, q\};\ p, q \in \mathbb{Z},\ |p - q| = 1\}$.

In the rest of this section, for every natural number $n > 0$, $\mathcal{B}_n \subseteq \mathcal{P}_n(\mathbb{Z}_2)$ will denote the n-partition of $\mathbb{Z}_2$ given as follows:

$\mathcal{B}_n = \{(x_i|\ i \leq n) \in \mathcal{P}_n(\mathbb{Z}_2);$ there exists an odd number $l \in \mathbb{Z}$ such that $x_i = ln + i$ for all $i \leq n$ or $x_i = ln - i$ for all $i \leq n\}$.

Thus, the paths belonging to $\mathcal{B}_n$ are just the arithmetic sequences $(x_i|\ i \leq n)$ of integers with the difference equal to 1 or -1 and with $x_0 = ln$ where $l \in \mathbb{Z}$ is an odd number. Note that each point of $z \in \mathbb{Z}$ belongs to at least one and at most two paths from $\mathcal{B}_n$. It belongs to two (different) paths from $\mathcal{B}_n$ if and only if there is $l \in \mathbb{Z}$ with $z = ln$ (in which case z is the first or last member of each of the two paths if l is odd or even, respectively).

Clearly, $u_{\mathcal{B}_n}$ is additive if and only if $n = 1$. The closure operator $u_{\mathcal{B}_1}$ coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k - 1, 2k, 2k + 1\};\ k \in \mathbb{Z}\}$ - cf. [8].

Theorem 3. *$(\mathbb{Z}, u_{\mathcal{B}_n})$ is a connected closure space.*

Proof. Put $D_l = \{ln + i;\ i \le n\}$ for each $l \in \mathbb{Z}$. Of course, D_l is connected in $(\mathbb{Z}, u_{\mathcal{B}_n})$ for every $l \in \mathbb{Z}$ (because $(ln + i;\ i \le n)$ is a $\mathcal{B}_n$-initial segment in $\mathbb{Z}_2$). It is also evident that $D_l \cup D_{l+1}$ is closed in $(\mathbb{Z}, u_{\mathcal{B}_n})$ whenever $l \in \mathbb{Z}$ is even.

Let ω denote the least infinite ordinal and let $(B_i|\ i < \omega)$ be the sequence given by $B_i = D_{\frac{i}{2}}$ whenever i is even and $B_i = D_{-\frac{i+1}{2}}$ whenever i is odd, i.e., $(B_i|\ i < \omega) = (D_0, D_{-1}, D_1, D_{-2}, D_2, ...)$. For each $l \in \mathbb{Z}$ there holds $D_l \cap D_{l+1} = \{(l+1)n)\} \ne \emptyset$. Thus, we have $B_0 \cap B_1 \ne \emptyset$. Let i_0 be a natural number with $i_0 > 1$. Then $B_{i_0} \cap B_{i_0 - 2} \ne \emptyset$ because $B_{i_0} = D_{\frac{i_0}{2}}$ and $B_{i_0-2} = D_{\frac{i_0}{2}-1}$ whenever i_0 is even, while $B_{i_0} = D_{-\frac{i_0+1}{2}}$ and $B_{i_0-2} = D_{-\frac{i_0+1}{2}+1}$ whenever i_0 is odd. Hence, $(\bigcup_{i<i_0} B_i) \cap B_{i_0} \ne \emptyset$ for each i_0, $0 < i_0 < \omega$. Therefore, $\bigcup_{i<\omega} B_i$ is connected. But $\bigcup_{i<\omega} B_i = \bigcup_{l\in\mathbb{Z}} D_l = \mathbb{Z}$, which proves the statement. $\square$

From now on, m will denote (similarly to n) a natural number with $m > 0$. Using results of the previous section, we may propose new structures on the digital spaces convenient for the study of digital images. Such a structure on $\mathbb{Z}^m$ is obtained as the product of m copies of the 2-adjacency graph $\mathbb{Z}_2$ with the n-partition given by the product of m-copies of the n-partition $\mathcal{B}_n$. More formally, we may consider the graph $G^m = \prod_{j=1}^m G_j$ on $\mathbb{Z}^m$, where $G_j = \mathbb{Z}_2$ for every $j \in \{1, 2, ..., m\}$, with the n-partition $\mathcal{B}_n^m \subseteq \mathcal{P}_n(G^m)$ given by $\mathcal{B}_n^m = \prod_{j=1}^m \mathcal{B}_j$ where $\mathcal{B}_j = \mathcal{B}_n$ for every $j \in \{1, 2, ..., m\}$. Of course, G^1 is the 2-adjacency graph on $\mathbb{Z}$ and G^2 and G^3 coincide with the 8-adjacency graph on $\mathbb{Z}^2$ and the 26-adjacency graph on $\mathbb{Z}^3$, i.e., the graph $(\mathbb{Z}^3, A_{26})$ where $A_{26} = \{\{(x_1, y_1, z_1), (x_2, y_2, z_2)\};\ (x_1, y_1, z_1), (x_2, y_2,\ z_2) \in \mathbb{Z}^3,\ \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\} = 1\}$.

Having defined the graphs G^m with n-partitions $\mathcal{B}_n^m$, we may study their topological properties with respect to the closure operators $u_{\mathcal{B}_n^m}$, i.e., we may investigate behavior of the closure spaces $\prod_{j=1}^m (\mathbb{Z}, u_{\mathcal{B}_n}) = (\mathbb{Z}^m, u_{\mathcal{B}_n^m})$. Such an investigation may be based on using results of the previous section concerning the products of closure operators on graphs induced by path partitions. In particular, as an immediate consequence of Theorems 2 and 3, we get:

Theorem 4. *$(\mathbb{Z}^m, u_{\mathcal{B}_n^m})$ is a connected closure space.*

Since the closure operator $u_{\mathcal{B}_n^m}$ coincides with the Khalimsky topology on $\mathbb{Z}^m$ for $n = 1$, we will suppose that $n > 1$ in the sequel. And we will restrict our considerations to $m = 2$ because this case is the most important one with respect to possible applications in digital topology. Thus, we will focuss on the closure spaces $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$.

We denote by $G(\mathcal{B}_n^2)$ the factor of the 8-adjacency graph on $\mathbb{Z}^2$ whose edges are those $\{(x_1, y_1), (x_2, y_2)\} \in A_8$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:

$$x_1 - y_1 = x_2 - y_2 = 2kn,$$
$$x_1 - y_1 = x_2 - y_2 = 2kn,$$
$$x_1 = x_2 = 2kn,$$
$$y_1 = y_2 = 2kn.$$

A section of the graph $G(\mathcal{B}_n^2)$ is demonstrated in Fig. 1 where only the vertices $(2kn, 2ln)$, $k, l \in \mathbb{Z}$, are marked out (by bold dots) and thus, on every edge drawn between two such vertices, there are $2n-1$ more (non-displayed) vertices, so that the edges represent $2n$ edges in the graph $G(\mathcal{B}_n^2)$. Clearly, every circle C in $G(\mathcal{B}_n^2)$ is a connected subset of the closure space $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ because it is a $\mathcal{B}_n^2$-circle in G^2. Indeed, C consists (i.e., is the union) of a finite sequence of paths from $\mathcal{B}_n^2$, hence $\mathcal{B}_n^2$-initial segments, such that every two consecutive paths have a point in common.

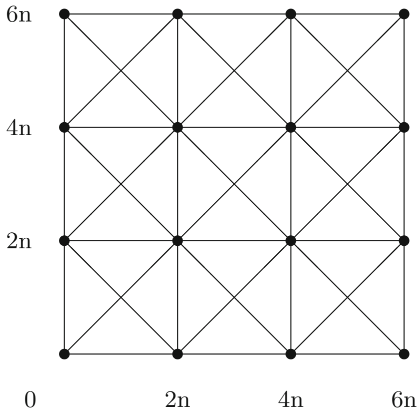


Fig. 1. A portion of the graph $G(\mathcal{B}_n^2)$

Definition 3. A circle J in the graph $G(\mathcal{B}_n^2)$ is said to be

(a) *fundamental* if, whenever $((2k+1)n, (2l+1)n) \in J$ for some $k, l \in \mathbb{Z}$, one of the following two conditions is true:
$\{((2k+1)n-1, (2l+1)n-1), (2k+1)n+1, (2l+1)n+1))\} \subseteq J$,
$\{((2k+1)n-1, (2l+1)n+1), (2k+1)n+1, (2l+1)n-1))\} \subseteq J$;

(b) a *Jordan curve* in $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ if the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ consists of two components.

Theorem 5. *Every fundamental circle in the graph $G(\mathcal{B}_n^2)$ is a Jordan curve in the closure space $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$.*

Proof. For every point $z = ((2k+1)n, (2l+1)n)$, $k, l \in \mathbb{Z}$, each of the following four subsets of $\mathbb{Z}^2$ will be called an n-fundamental triangle (given by z):

$$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \le x + 2ln - 2kn\},$$
$$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \ge 4ln + 2kn - x\},$$
$$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \ge x + 2ln - 2kn\},$$
$$\{(r,s) \in \mathbb{Z}^2;\ 2kn \le r \le (2k+2)n,\ 2ln \le s \le (2l+2)n,\ y \le 4ln + 2kn - x\}.$$

The points of any n-fundamental triangle form a segment of the shape of a (digital) rectangular triangle. Obviously, in each of the four n-fundamental triangles given by z, z is the middle point of the hypotenuse of the triangle. Clearly, the edges of any n-fundamental triangle form a circle in the graph $G(\mathcal{B}_n^2)$, hence a $\mathcal{B}_n^2$-circle in G^2. It may easily be seen that every n-fundamental triangle is connected in $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ and so is also every set obtained from an n-fundamental triangle by subtracting some of its edges. We will say that a (finite or infinite) sequence S of n-fundamental triangles is a tiling sequence if the members of S are pairwise different and every member of S, excluding the first one, has an edge in common with at least one of its predecessors. Given a tiling sequence S of n-fundamental triangles, we denote by S' the sequence obtained from S by subtracting, from every member of the sequence, those of its edges that are not shared with any other member of the sequence. It immediately follows that, for every tiling sequence S of n-fundamental triangles, the set $\bigcup\{T;\ T \in S\}$ is connected in $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ and the same is true for the set $\bigcup\{T;\ T \in S'\}$.

Let J be a fundamental circle in the graph $G(\mathcal{B}_n^2)$. Then J constitutes the border of a polygon $S_F \subseteq \mathbb{Z}^2$ consisting of n-fundamental triangles. More precisely, S_F is the union of some n-fundamental triangles such that any pair of them is disjoint or meets in just one (common) edge. Let U be a (finite) tiling sequence of the n-fundamental triangles contained in S_F. Then we have $S_F = \bigcup\{T;\ T \in U\}$. Since every n-fundamental triangle $T \in U$ is connected, S_F is connected, too. Similarly, U' is a (finite) sequence with $S_F - J = \bigcup\{T;\ T \in U'\}$ and, since every member of U' is connected, $S_F - J$ is connected, too.

Further, let V be an (infinite) tiling sequence of the n-fundamental triangles which are not contained in S_F. Put $S_I = \bigcup\{T;\ T \in V\}$. Since every n-fundamental triangle $T \in V$ is connected, S_I is connected, too. Similarly, V' is an (infinite) sequence with $S_I - J = \bigcup\{T;\ T \in V'\}$ and, since every member of V' is connected, $S_I - J$ is connected, too.

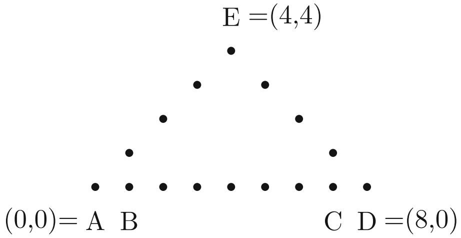
It may easily be seen that every $\mathcal{B}_n^2$-walk $C = (z_i|\ i \leq k)$, $k > 0$ a natural number, in the 8-adjacency graph G^2 on $\mathbb{Z}^2$ connecting a point of $S_F - J$ with a point of $S_I - J$ meets J (i.e., meets an edge of an n-fundamental triangle which is contained in J). Therefore, the set $\mathbb{Z}^2 - J = (S_F - J) \cup (S_I - J)$ is not connected in $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$. We have shown that $S_F - J$ and $S_I - J$ are components of the subspace $\mathbb{Z}^2 - J$ of $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$. $\square$

Remark 1. (a) If C is a fundamental circle in $G(\mathcal{B}_n^2)$, then, by the proof of Theorem 5, one of the components of $\mathbb{Z}^2 - C$ is finite (the inside component $S_F - J$) and the other is infinite (the outside component $S_I - J$).

(b) The famous Jordan curve theorem proved in [5] states that every circle in the graph $G(\mathcal{B}_1^2)$ having at least four points and containing, with each if its points, just two points adjacent to it is a Jordan curve in the Khalimsky topological space $(\mathbb{Z}^2, u_{\mathcal{B}_1^2})$.

(c) A digital analogue of the Jordan curve theorem similar to Theorem 5 may be found in [12] where a different, quite complicated procedure is used to prove it based on employing Jordan curves in the Khalimsky topology on $\mathbb{Z}^2$ and quotient closure spaces.

Example 1. Consider the following (digital picture of a) triangle:



While the triangle ADE is a Jordan curve in $(\mathbb{Z}^2, u_{\mathcal{B}_2^2})$, it is not a Jordan curve in the Khalimsky topological space $(\mathbb{Z}^2, u_{\mathcal{B}_1^2})$. In order that this triangle be a Jordan curve in the Khalimsky topological space, we have to delete the points A,B,C and D. But this will lead to a considerable deformation of the triangle.

5 Conclusion

We introduced and investigated the concept of path-set induced closure operators on graphs. It was shown that connectedness with respect to these closure operators is preserved by a special product of the operators. We applied this result to the products of pairs of copies of the closure operators induced on a graph with the vertex set $\mathbb{Z}$ by certain path partitions. In this way, we obtained closure operators $u_{\mathcal{B}_n^2}$ ($n > 1$ a natural number) on the digital plane $\mathbb{Z}^2$ with the closure spaces $(\mathbb{Z}^2, u_{\mathcal{B}_n^2})$ connected that were then discussed.

One of the basic problems of digital topology is to find structures on the digital plane $\mathbb{Z}^2$ convenient for the study of digital images. A basic criterion of such a convenience is the validity of a digital analogue of the Jordan curve theorem. Namely, in digital images, which are regarded as approximations of real ones, digital versions of simple closed curves form borders of objects. To avoid undesirable paradoxes, it is necessary that the curves (circles) satisfy a digital analogue of the Jordan curve theorem, i.e., separate the digital plane into precisely two components - the inside and the outside of an object. The above Theorem 5 provides such a digital analogue of the Jordan curve theorem, thus making possible to use the closure operators $u_{\mathcal{B}_n^2}$ as convenient background structures on the digital plane $\mathbb{Z}^2$ for the study of digital images. They may also be useful for solving problems of digital image processing closely related to connectedness like pattern recognition, border detection, contour filling, etc.

Acknowledgement. This work was supported by Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4Innovations Excellence in Science - LQ1602".

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