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Research Article

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Path-induced closure operators on graphs for defining digital Jordan surfaces

Abstract: Given a simple graph with the vertex set X, we discuss a closure operator on X induced by a set of paths with identical lengths in the graph. We introduce a certain set of paths of the same length in the 2-adjacency graph on the digital line $\mathbb{Z}$ and consider the closure operators on $\mathbb{Z}^m$ (m a positive integer) that are induced by a special product of m copies of the introduced set of paths. We focus on the case $m = 3$ and show that the closure operator considered provides the digital space $\mathbb{Z}^3$ with a connectedness that may be used for defining digital surfaces satisfying a Jordan surface theorem.

Keywords: simple graph, path, closure operator, connectedness, digital space, digital surface, Khalimsky topology, Jordan surface theorem

MSC: 54A05, 54D05, 52C99, 68R10

1 Introduction

In digital picture analysis, detection of object borders plays an important role in solving numerous problems such as pattern recognition etc. - cf. [1]. In two-dimensional digital pictures, it is required that object borders be digital Jordan curves, i.e., subsets of the digital plane $\mathbb{Z}^2$ satisfying a digital analog of the Jordan curve theorem. (Recall that the classical Jordan curve theorem states that a simple closed curve separates the Euclidean plane into precisely two connected components). It is, therefore, necessary to equip the digital plane with a connectedness structure making it possible to define digital Jordan curves. In the classical approach to this problem, a pair of adjacency relations (4- and 8-adjacency) on $\mathbb{Z}^2$ is employed (see [2, 3]). In [4], a new, topological approach was proposed using a single connectedness structure, the so-called Khalimsky topology to provide $\mathbb{Z}^2$ with a connectedness structure. The topological approach was then developed by many authors, see, e.g., [5–8].

In three-dimensional pictures, object borders are to be digital surfaces, i.e., subsets of the digital space $\mathbb{Z}^3$ satisfying a digital analog of the Jordan surface theorem (which is also known as the Jordan-Brouwer theorem). A classical approach to this three-dimensional problem is based, like in the two-dimensional case, on using a pair of adjacency relations (6- and 26-adjacency) on $\mathbb{Z}^3$- see [9–12]. The topological approach employing the Khalimsky topology on $\mathbb{Z}^3$ was applied, e.g., in [13, 14].

The present paper is a contribution to the topological approach to the problem of recognizing digital surfaces in $\mathbb{Z}^2$. Instead of the Khalimsky topology, we employ closure operators that are induced by a set of paths of the same length in a simple graph. We introduce, for every positive integer n, a certain set of paths of identical lengths n in the 2-adjacency graph on the digital line $\mathbb{Z}$. For every positive integer m, we obtain a closure operator on $\mathbb{Z}^m$ induced by a special product of m copies of the introduced set of paths in $\mathbb{Z}$. For

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$n = 2$, we get the well-known Khalimsky topology on $\mathbb{Z}^m$. We focus on the case of $n = m = 3$ and, for the obtained closure operator on $\mathbb{Z}^3$, we prove a digital Jordan surface theorem.

Sets of paths of identical lengths in a graph were proposed for the study of connectedness in digital spaces in [15]. Closure operators induced by such sets of paths were used in [16] for proving a digital Jordan curve theorem. In [17], correspondences between sets of paths and closure operators in a simple graph were studied. In the present paper, we build on the concepts and results in [17]. To make the paper self-contained, we repeat some of them.

By a *graph* $G = (V, E)$, we understand an (undirected simple) graph (without loops) where $V \neq \emptyset$ is the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ is the set of *edges* of G. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. Recall that a *walk* in G is a (finite) sequence $(x_i|\ i \leq n)$, i.e., $(x_0, x_1, \ldots, x_n)$, of pairwise different vertices of V such that x_i is adjacent to x_{i+1} whenever $i < n$. If, moreover, the members of $(x_i|\ i \leq n)$ are pairwise different, then $(x_i|\ i \leq n)$ is called a *path*. The non-negative integer n is called the *length* of the walk (path) $(x_i|\ i \leq n)$. A sequence $(x_i|\ i \leq n)$ of vertices of G is called a *circle* if $n > 2$, $x_0 = x_n$, and $(x_i|\ i < n)$ is a path. For the graph-theoretic background, we refer to [18].

Given graphs $G_j = (V_j, E_j,)$, $j = 1, 2, \ldots, m$ ($m > 0$ an integer), we define their *strong product* to be the graph $\prod_{j=1}^m G_j = (\prod_{j=1}^m V_j, E)$ with the set of edges $E = \{\{(x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m)\};$ there exists a nonempty subset $J \subseteq \{1, 2, \ldots, m\}$ such that $\{x_j, y_j\} \in E_j$ for every $j \in J$ and $x_j = y_j$ for every $j \in \{1, 2, \ldots, m\} - J\}$. Note that the strong product differs from the cartesian product of G_j, $j = 1, 2, \ldots, m$, i.e., from the graph $(\prod_{j=1}^m V_j, F)$ where $F = \{\{(x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_m)\};\ \{x_j, y_j\} \in E_j$ for every $j \in \{1, 2, \ldots, m\}\}$. More precisely, we have $F \subsetneq E$ whenever $m > 1$. The strong product of a pair of graphs coincides with that introduced in [19].

If $G_j = G$ for every $j = 1, 2, \ldots, m$, we write G^m instead of $\prod_{j=1}^m G_j$.

By a *closure operator* u on a set X, we mean a map $u: \exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is
(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).
The pair (X, u) is then called a *closure space*.

A closure operator u on X that is
(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$)
is called a *Kuratowski closure operator* or a *topology* and the pair (X, u) is called a *topological space*.

Given a cardinal $m > 1$, a closure operator u on a set X and the closure space (X, u) are called an S_m-*closure operator* and an S_m-*closure space* (briefly, an S_m-*space*), respectively, if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \operatorname{card} B < m\}$.

S_2-topologies (S_2-topological spaces) are usually called *Alexandroff topologies* (*Alexandroff spaces*) - see [7]. Similarly to [17], we will use some basic topological concepts such as closed subsets, subspaces, connected subsets, (connected) components etc. (see, e.g., [20]) naturally extended from topological spaces to closure ones. The behavior of extended concepts is then analogous to that of the original ones. In particular, we will employ the fact that the union of a (finite or infinite) sequence of connected subsets of a closure space is connected in the space if every pair of consecutive members of the sequence has a nonempty intersection.

We will say that a subset Y of a closure space (X, u) *separates the space into exactly two components* if the subspace $X - Y$ of (X, u) has exactly two components.

2 Closure operators induced by sets of paths

In the sequel, n will denote a positive integer. Given a graph G, we denote by $\mathcal{P}_n(G)$ the set of all paths of length n in G. For every set of paths (path set for short) $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we put
$\hat{\mathcal{B}} = \{(x_i| i \le m) \in \mathcal{P}_m(G); \ 0 < m \le n$ and there exists $(y_i| i \le n) \in \mathcal{B}$ such that $x_i = y_i$ for every $i \le m\}$ and
$\mathcal{B}^\star = \{(x_i| i \le m) \in \mathcal{P}_m(G); \ 0 < m \le n$ and $(x_i| i \le m) \in \hat{\mathcal{B}}$ or $(x_{m-i}| i \le m) \in \hat{\mathcal{B}}\}$.
Thus, we have $\mathcal{B} \subseteq \hat{\mathcal{B}} \subseteq \mathcal{B}^\star$.

Let G_j be a graph and $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$ for every $j = 1, 2, ..., m$ ($m > 0$ an integer). Then, we put $\prod_{j=1}^{m} \mathcal{B}_j = \{((x_i^1, x_i^2, ..., x_i^m)| i \le n);$ there is a nonempty subset $J \subseteq \{1, 2, ..., m\}$ such that $(x_i^j| i \le n) \in \mathcal{B}_j$ for every $j \in J$ and $(x_i^j| i < j)$ is a constant sequence for every $j \in \{1, 2, ..., m\} - J\}$. It is evident that $\prod_{j=1}^{m} \mathcal{B}_j \subseteq \mathcal{P}_n(\prod_{j=1}^{m} G_j)$. $\prod_{j=1}^{m} \mathcal{B}_j$ will be called the *strong product* of $\mathcal{B}_j$, $j = 1, 2, ..., m$ (it will always be clear whether a strong product discussed relates to graphs or path sets). If $G_j = G$ and $\mathcal{B}_j = \mathcal{B}$ for every $j = 1, 2, ..., m$, we write $\mathcal{B}^m$ instead of $\prod_{j=1}^{m} \mathcal{B}_j$.

Let G be a graph with the vertex set V and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. For every $X \subseteq V$, we put

$$f_n(\mathcal{B})X = X \cup \{x \in V; \ \text{there exists } (x_i| i \le m) \in \hat{\mathcal{B}} \text{ with } \{x_i; \ i < m\} \subseteq X \text{ and } x_m = x\}.$$

It may easily be seen that $f_n(\mathcal{B})$ is an S_{n+1}-closure operator on V - it will be said to be *associated* with $\mathcal{B}$. It is evident that every path belonging to $\mathcal{B}^\star$ is a connected subset of the closure space $(V, f_n(\mathcal{B}))$. For the properties of the closure operators $f_n(\mathcal{B})$ see [17].

Recall [16] that, given a graph $G = (V, E)$ and $\mathcal{B} \subseteq \mathcal{P}_n(G)$, a sequence $C = (x_i| i \le p)$, $p > 0$ an integer, of vertices of V is called a $\mathcal{B}$-*walk* in G if there is an increasing sequence $(i_k| k \le q)$, $q > 0$ an integer, of non-negative integers with $i_0 = 0$ and $i_q = p$ such that $i_k - i_{k-1} \le n$ and $(x_i| i_{k-1} \le i \le i_k) \in \mathcal{B}^\star$ for every k with $0 < k \le q$. If the terms of C are pairwise different, then C is called a $\mathcal{B}$-*path* in G.

A $\mathcal{B}$-walk C is said to be a $\mathcal{B}$-*circle* if, for every pair i_0, i_1 of different integers with $0 \le i_0, i_1 \le p$, $x_{i_0} = x_{i_1}$ is equivalent to $\{i_0, i_1\} = \{0, p\}$.

Clearly, every $\mathcal{B}$-walk ($\mathcal{B}$-path, $\mathcal{B}$-circle) in a graph $G = (V, E)$ is a walk (path, circle) in G and both concepts coincide if $\mathcal{B} = \{(x, y); \ \{x, y\} \in E\} \subseteq \mathcal{P}_1(G)$.

We will need the following statement proved in [16]:

Proposition 2.1. *Let G be a graph with the vertex set V and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. A subset $A \subseteq V$ is connected in the closure space $(V, f_n(\mathcal{B}))$ if and only if any two different vertices of G belonging to A can be joined by a $\mathcal{B}$-walk in G contained in A.*

3 The closure operator on $\mathbb{Z}^3$ induced by a set of paths of length 2

Recall that the 2-adjacency graph (on $\mathbb{Z}$) is the graph $H = (\mathbb{Z}, A_2)$ where $A_2 = \{\{p, q\}; \ p, q \in \mathbb{Z}, \ |p - q| = 1\}$.

For every $l \in \mathbb{Z}$, we put

$$I_l = \begin{cases} (ln + i| i \le n) \text{ if } l \text{ is odd}, \\ ((l + 1)n - i| i \le n) \text{ if } l \text{ is even}. \end{cases}$$

In the sequel, (for a given integer $n > 1$) $\mathcal{B}$ will denote the set $\mathcal{B} \subseteq \mathcal{P}_n(H)$ given by $\mathcal{B} = \{I_l; \ l \in \mathbb{Z}\}$. Thus, all paths I_l belonging to $\mathcal{B}$ are just the arithmetic sequences $(x_i|i \le n)$ of integers where the difference equals 1 and $x_0 = ln$ if l is odd and the difference equals 1 and $x_0 = (l + 1)n$ if l is even. Note that each element $z \in \mathbb{Z}$ belongs to at least one and at most two paths in $\mathcal{B}$. It belongs to two (different) paths from $\mathcal{B}$ if and only if there is $l \in \mathbb{Z}$ with $z = ln$ (in which case, z is the first member of each of the paths I_l and I_{l-1} if l is odd, and z is the last member of each of the two paths if l is even).

The closure space $(\mathbb{Z}^m, f_1(\mathcal{B}^m))$ coincides with the m-dimensional Khalimsky space for every positive integer m. A digital Jordan curve theorem for the Khalimsky plane $(\mathbb{Z}^2, f_1(\mathcal{B}^2))$ was proved in [4] and a digital Jordan surface theorem for the Khalimsky space $(\mathbb{Z}^3, f_1(\mathcal{B}^3))$ was proved in [13]. In [16], a Jordan curve theorem
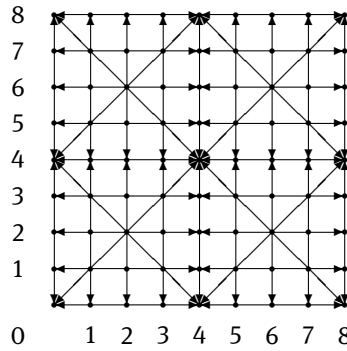
Figure 1: The path set $\mathcal{B}^2 \subseteq \mathcal{P}_2(H^2)$.

for the closure space $(\mathbb{Z}^2, f_n(\mathcal{B}^2))$ is proved (for an arbitrary integer $n > 1$). In the present note, we will focus on proving a digital Jordan surface theorem for the closure space $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$. Note that the graphs H^2 and H^3 are simply the well known 8- and 26-adjacency graphs, respectively - cf. [11]. The path set $\mathcal{B}^2 \subseteq \mathcal{P}_2(H^2)$ is demonstrated in Figure 1 where the paths belonging to $\mathcal{B}^2$ are marked by line segments directed from the first to the last terms of the paths.

From now on, we assume that $n = 2$. Hence, $\mathcal{B}$ is a set of paths of length 2 ($\mathcal{B} \subseteq \mathcal{P}_2(H)$) and so is $\mathcal{B}^3$ ($\mathcal{B}^3 \subseteq \mathcal{P}_2(H^3)$). By [16], $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$ is connected.

Definition 3.1. Each of the following subsets of $\mathbb{Z}^3$ will be called a *fundamental rectangle*:

(1) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ z = 4m\},\ k, l, m \in \mathbb{Z}$,
(2) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4k,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$,
(3) $\{(x, y, z) \in \mathbb{Z}^3;\ x = 4k,\ 4l \le y \le 4l + 4, 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$,
(4) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = x - 4k + 4l,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$,
(5) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4k + 4l + 4 - x,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$,
(6) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4, z = x - 4k + 4m\},\ k, l, m \in \mathbb{Z}$,
(7) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ z = 4k + 4m + 4 - x\},\ k, l, m \in \mathbb{Z}$,
(8) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ z = y - 4l + 4m\},\ k, l, m \in \mathbb{Z}$,
(9) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ z = 4l + 4m + 4 - y\},\ k, l, m \in \mathbb{Z}$.

Clearly, a subset $T \subseteq \mathbb{Z}^3$ is a fundamental rectangle if and only if there is a digital cube $\{(x, y, z);\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$, such that T is a (digital) face of the cube or is the intersection of the cube with the (digital) plane that is perpendicular to a face of the cube and contains one of the two (digital) diagonals of the face. Hence, every fundamental rectangle T consists of 25 points and has the form of a digital square parallel to a coordinate plane (hence perpendicular to the other two coordinate planes) or a digital rectangle perpendicular to a coordinate plane with the angle $\frac{\pi}{4}$ between T and each of the other two coordinate planes. Thus, it is clear which sets of points (digital line segments) are the sides of a fundamental rectangle. By the help of Figure 1 (where the path set $\mathcal{B}^2$ demonstrated is a two-dimensional projection of $\mathcal{B}^3$), we may easily see that every two different points of a fundamental rectangle may be joined by a $\mathcal{B}^3$-path contained in the rectangle. Thus, by Proposition 2.1, every fundamental rectangle is connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$.

Definition 3.2. Each of the following subsets of $\mathbb{Z}^3$ will be called a *fundamental triangle*:

(1) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le x - 4k + 4l,\ z = 4m\},\ k, l, m \in \mathbb{Z}$,
(2) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ x - 4k + 4l \le y \le 4l + 4,\ z = 4m\},\ k, l, m \in \mathbb{Z}$,
(3) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4k + 4l + 4 - x,\ z = 4m\},\ k, l, m \in \mathbb{Z}$,
(4) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4k + 4l + 4 - x \le y \le 4l + 4,\ z = 4m\},\ k, l, m \in \mathbb{Z}$,
(5) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4l, 4m \le z \le x - 4k + 4m\},\ k, l, m \in \mathbb{Z}$,

(6) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4l,\ x - 4k + 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(7) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4l,\ 4m \le z \le 4k + 4m + 4 - x\},\ k, l, m \in \mathbb{Z},$

(8) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ y = 4l,\ 4k + 4m + 4 - x \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(9) $\{(x, y, z) \in \mathbb{Z}^3;\ x = 4k,\ 4l \le y \le 4l + 4, 4m \le z \le y - 4l + 4m\},\ k, l, m \in \mathbb{Z},$

(10) $\{(x, y, z) \in \mathbb{Z}^3;\ x = 4k,\ 4l \le y \le 4l + 4,\ y - 4l + 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(11) $\{(x, y, z) \in \mathbb{Z}^3;\ x = 4k,\ 4l \le y \le 4l + 4,\ 4m \le z \le 4l + 4m + 4 - y\},\ k, l, m \in \mathbb{Z},$

(12) $\{(x, y, z) \in \mathbb{Z}^3;\ x = 4k,\ 4l \le y \le 4l + 4\ 4l + 4m + 4 - y \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}.$

Clearly, a subset $T \subseteq \mathbb{Z}^3$ is a fundamental triangle if and only if there is a fundamental rectangle parallel to a coordinate plane, hence a square, such that T is one of the four (digital) half-square triangles obtained from the square (by splitting it along a diagonal). Thus, every fundamental triangle has the form of a (digital) isosceles right-angled triangle parallel to a coordinate plane. It is, therefore, clear which sets of points (digital line segments) are the sides of a fundamental triangle. As in the case of fundamental rectangles, fundamental triangles, too, may easily be seen, by the help of Figure 1 and Proposition 2.1, to be connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$.

Definition 3.3. Each of the following subsets of $\mathbb{Z}^3$ will be called a *fundamental prism*:

(1) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le x - 4k + 4l,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(2) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ x - 4k + 4l \le y \le 4l + 4,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(3) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4k + 4l + 4 - x,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(4) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4k + 4l + 4 - x \le y \le 4l + 4,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(5) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4, 4m \le z \le x - 4k + 4m\},\ k, l, m \in \mathbb{Z},$

(6) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ x - 4k + 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(7) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ 4m \le z \le 4k + 4m + 4 - x\},\ k, l, m \in \mathbb{Z},$

(8) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4\ 4k + 4m + 4 - x \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(9) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ 4m \le z \le y - 4l + 4m\},\ k, l, m \in \mathbb{Z},$

(10) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ y - 4l + 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z},$

(11) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ 4m \le z \le 4l + 4m + 4 - y\},\ k, l, m \in \mathbb{Z},$

(12) $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4\ 4l + 4m + 4 - y \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}.$

Clearly, every fundamental prism has the form of a digital triangular prism with 75 points such that each of its faces is a fundamental triangle or a fundamental rectangle. Every cube $\{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le 4l + 4,\ 4m \le z \le 4m + 4\},\ k, l, m \in \mathbb{Z}$, is the union of two different fundamental prisms having a face in common (and all fundamental prisms are obtained in this way).

Lemma 3.4. *Every fundamental prism is connected in* $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$ *and so is every subset of* $\mathbb{Z}^3$ *obtained from a fundamental prism by removing some of its faces.*

Proof. Let P be the fundamental prism given by condition (1) in Definition 3.3, i.e., $P = \{(x, y, z) \in \mathbb{Z}^3;\ 4k \le x \le 4k + 4,\ 4l \le y \le x - 4k + 4l,\ 4m \le z \le 4m + 4\}$ where $k = l = m = 0$. Clearly, P is the union of the five triangles obtained as the intersection of P with the digital plane (parallel to the coordinate plane xy) $\{(x, y, z) \in \mathbb{Z}^3;\ z = k\}$ where k is one of the integers 0,1,2,3,4. The projection of each of the five triangles onto the coordinate plane xy is the fundamental triangle $(0,0)(4,0)(4,4)$ in Figure 1 (which is connected in $(\mathbb{Z}^2, f_2(\mathcal{B}^2))$) by Proposition 2.1). Hence, each of the five triangles is connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$. The same is true for every set obtained from any of the five triangles by removing some of its edges. Put $C = ((3, 1, 0), (3, 1, 1), (3, 1, 2), (3, 1, 3), (3, 1, 4))$. Since both $((3, 1, 2), (3, 1, 1), (3, 1, 0))$ and $((3, 1, 2), (3, 1, 3), (3, 1, 4))$ belong to $\mathcal{B}^3$, C is an $\mathcal{B}^3$-path. Thus, by Proposition 2.1, C is a connected subset of $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$. It is evident that C meets each of the five triangles and is contained in their union and the same is true even if some sides of the triangles are removed. Therefore, the fundamental prism P (which is the union of the triangles) is connected and the same is true if some faces of the prism are removed. If some of the integers k, l, m differ from 0, the proof is much the same. Thus, the Lemma is proved for fundamental prisms of the form (1) in Definition 3.3. For fundamental prisms of the other eleven forms, the proofs are done along similar lines. $\square$
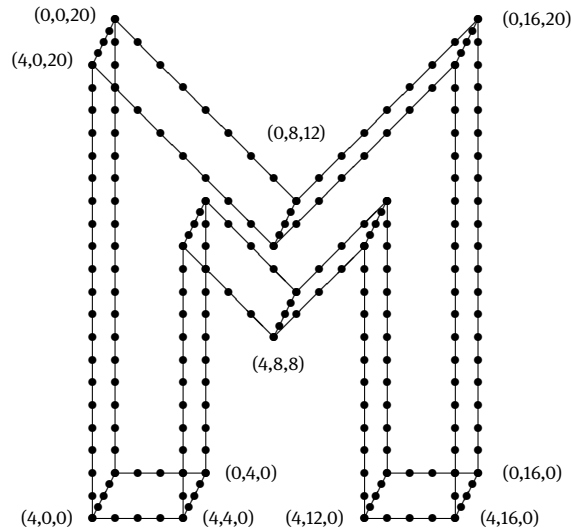
Figure 2: A Jordan surface in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$.

Theorem 3.5. (Digital Jordan Surface Theorem) *Let S be a connected subset of $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$ which is the union of a finite (non-empty) set $\mathcal{F}$ of fundamental triangles and fundamental rectangles such that*

(1) *For every pair $F_1, F_2 \in \mathcal{F}$, $F_1 \neq F_2$, we have* $\mathrm{card}(F_1 \cap F_2) \leq 1$ *or $F_1 \cap F_2 = J$ where J is a common side of F_1 and F_2.*

(2) *For every $F_1 \in \mathcal{F}$, if J is a side of F_1, then there exists exactly one $F_2 \in \mathcal{F}$, $F_1 \neq F_2$, such that J is a a side of F_2.*

Then, S separates $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$ into exactly two components and the union of S with each of them is connected.

Proof. Let *S* satisfy the conditions of the statement. Clearly, *S* is a polyhedral surface which is the union of all faces of a polyhedron $T_F \subseteq \mathbb{Z}^3$ consisting of fundamental prisms. More precisely, T_F may be expressed as the union of a (finite) sequence of pairwise different fundamental prisms such that any two of them are disjoint or meet in just one face in common and every prism in the sequence, except for the first one, has a face in common with at least one of its predecessors. However, the set $T_I = (\mathbb{Z}^3 - T_F) \cup S$, too, may be written as the union of such an (infinite) sequence of fundamental prisms. By Lemma 3.4, T_F, $T_F - S$, T_I, and $T_I - S$ are connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$.

It is obvious that every $\mathcal{B}^3$-walk $C = (z_i | i \leq k)$, $k > 0$ an integer, joining a point of $T_F - S$ with a point of $T_I - S$ meets *S* (i.e., meets a fundamental prism face contained in *S*). Thus, by Proposition 2.1, the set $\mathbb{Z}^3 - S = (T_F - S) \cup (T_I - S)$ is not connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$. Hence, $T_F - S$ and $T_I - S$ are components of the subspace $\mathbb{Z}^3 - S$ of $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$, $T_F - S$ finite and $T_I - S$ infinite, with T_F and T_I connected in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$. □

Conclusion. We have introduced a closure operator on the digital space $\mathbb{Z}^3, f_2(\mathcal{B}^3)$, which provides a connectedness that allows for a digital Jordan surface theorem (Theorem 3.5). The Jordan surfaces introduced, i.e., the surfaces *S* satisfying the assumptions of Theorem 3.5, have the advantage over the Jordan surfaces with respect to the Khalimsky topology proposed in [13] that the angle between a pair of fundamental rectangles belonging to *S* may be $\frac{\pi}{4}$. For example, the surface of the letter M demonstrated in Figure 2 (where only the points on the edges of the letter are marked) is a Jordan surface in $(\mathbb{Z}^3, f_2(\mathcal{B}^3))$ but it is not a Jordan surface with respect to the Khalimsky topology $f_1(\mathcal{B}^3)$ in the sense of [13] because there are four pairs of fundamental rectangles that meet at an angle of $\frac{\pi}{4}$. Therefore, the closure operator $f_2(\mathcal{B}^3)$ gives a convenient structure on $\mathbb{Z}^3$ for the study of three-dimensional digital images providing more flexible digital Jordan surfaces than the Khalimsky topology.

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