Research Article

Characterization of Uninorms on Bounded Lattices and Pre-order They Induce

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ABSTRACT

In Hliněná *et al.*, Pre-orders and orders generated by uninorms, in 15th International Conference IPMU 2014, Proceedings, Part III, Montpellier, France, 2014, pp. 307–316 the authors, inspired by Karaçal and Kesicioğlu, A t-partial order obtained from t-norms, Kybernetika. 47 (2011), 300–314, introduced a pre-order induced by uninorms. This contribution is devoted to a classification of families of uninorms by means of pre-orders (and orders) they induce. Philosophically, the paper follows the original idea of Clifford, Naturally totally ordered commutative semigroups, Am. J. Math. 76 (1954), 631–646. The present paper is an extension of the paper Hliněná and Kalina, A characterization of uninorms by means of a pre-order they induce, in Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019), Atlantis Press, 2019, pp. 595–601 that was presented at EUSFLAT 2019. As a by-product, we present a t-norm that possesses a single discontinuity point.

1. INTRODUCTION

In this paper we study pre-orders induced by uninorms on bounded lattices. The main idea is based on that of Karaçal and Kesicioğlu [1], and follows the original idea of Clifford [2] and Mitsch [3]. The main idea of the authors of this paper is to show a relationship between families of uninorms and families of pre-orders (partial orders, in some cases) they induce (see Hliněná *et al.* [4]). Another relation induced by uninorms, that is always a partial order (see Definition 16), was proposed by Ertuğrul *et al.* [5]. Here, the main intention of the authors was to get a partial order. This relation (partial order) does not suit well our purposes. In Preliminaries, we will explain what can be characterized by particular types of (pre-)orders.

The present paper is an extension of Hliněná and Kalina [6] that was presented at the EUSFLAT 2019 conference held in Prague.

2. PRELIMINARIES

We assume that readers are familiar with bounded lattices. For information on this topic we recommend the monograph by Birkhoff [7].

In the whole paper, $(L, \leqslant_L, \mathbf{0}_L, \mathbf{1}_L)$ will denote a (not fixed) bounded lattice, where L is the set of all values of the lattice. If no confusion may occur, by L we will denote also the whole bounded lattice. For arbitrary $x, y \in L$, if these elements are incomparable, we will denote the fact by

$$x \parallel_L y.$$

The set of all elements which are incomparable with x, will be denoted by I_x, i.e.,

$$I_x = \{y \in L : y \parallel_L x\}. \tag{1}$$

In this section we review some well-known types of monotone commutative monoidal operations on L and provide an overview of, from the point of view of this contribution, important steps in introducing orders (and pre-orders) induced by semigroups. Before starting the review of the well-known monoidal operations we introduce yet one notation.

For a function $F : A \to B$, where A and B are some non-void sets, and a set C with $\varnothing \neq C \subset A$, the restriction of F to C will be denoted by

$$F \upharpoonright C.$$

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2.1. Known Types of Monotone Commutative Monoidal Operations on [0, 1] and on *L*

In this part we give just very brief review of well-known types of monotone commutative monoidal operations on *L*. For more details on monoidal operations on [0, 1] we recommend monographs Calvo *et al.* [8] and Klement *et al.* [9].

Definition 1 (see, [9]) A *triangular norm T* (t-norm for short) on [0, 1] is a commutative, associative, monotone binary operation, fulfilling the boundary condition $T(x, 1) = x$, for all $x \in [0, 1]$.

Definition 2 (see, [9]) A *triangular conorm S* (t-conorm for short) on [0, 1] is a commutative, associative, monotone binary operation, fulfilling the boundary condition $S(x, 0) = x$, for all $x \in [0, 1]$.

Definition 3 Let *L* be a bounded lattice. A function $N : L \to L$ is a *negation* if

- $N(\mathbf{0}_L) = \mathbf{1}_L, N(\mathbf{1}_L) = \mathbf{0}_L$,

- *N* is monotone (decreasing).

If moreover *N* is a bijection, *N* is said to be *strict*. If $N(N(x)) = x$ for all $x \in L$, *N* is said to be *strong*.

Remark 1.

(a) If *T* is a t-norm, then

$$S(x, y) = 1 - T(1 - x, 1 - y)$$

is a t-conorm and vice versa. We obtain a dual pair (T, S) of a t-norm and a t-conorm.

(b) t-norms and t-conorms on bounded lattices are defined in the same way as on [0, 1]. Concerning their mutual relationship (duality), if *L* is a lattice with a strong negation *N*, for every t-norm *T*

$$S(x, y) = N(T(N(x), N(y)))$$

is the dual t-conorm, and vice versa.

Example 1. Well-known examples of t-norms and their dual t-conorms are the following:

- $T_M(x, y) = \min(x, y), \quad S_M(x, y) = \max(x, y)$,

- $T_P(x, y) = x.y, \quad S_P(x, y) = x + y - x.y$,

- $T_L(x, y) = \max(x + y - 1, 0), \quad S_L(x, y) = \min(x + y, 1)$.

Casasnovas and Mayor [10] introduced divisible t-norms.

Definition 4 (see [10]) Let *L* be a bounded lattice and $T : L \times L \to L$ be a t-norm. *T* is said to be *divisible* if the following condition is satisfied for all $(x, y) \in L^2$

$$(x \leqslant y) \Rightarrow (\exists z \in L)(T(y, z) = x). \tag{2}$$

Of course, a t-norm $T : [0, 1]^2 \to [0, 1]$ is divisible if and only if it is continuous.

Definition 5 (see, [8]) Let $* : L^2 \to L$ be a binary commutative operation. Then

i. element *c* is said to be *idempotent* if $c * c = c$,

ii. element *e* is said to be *neutral* if $e * x = x$ for all $x \in L$,

iii. element *a* is said to be *annihilator* if $a * x = a$ for all $x \in L$.

Definition 6 (see [11]) A *uninorm U* is a function $U : [0, 1]^2 \to [0, 1]$ that is increasing, commutative, associative and has a neutral element $e \in [0, 1]$.

Karaçal and Mesiar [12] have shown that on every bounded lattice *L* possessing at least three elements we can choose an element $e \notin \{\mathbf{0}_L, \mathbf{1}_L\}$ and construct a uninorm $U : L^2 \to L$ with the neutral element *e*.

Remark 2. Let *L* be a bounded lattice. For any uninorm *U* with neutral element equal to *e* we denote

$$A(e) = [\mathbf{0}_L, e[\times]e, \mathbf{1}_L] \ \cup \ ]e, \mathbf{1}_L] \times [\mathbf{0}_L, e[. \tag{3}$$

- If $e \notin \{\mathbf{0}_L, \mathbf{1}_L\}$ is the neutral element of *U*, we say that *U* is a *proper uninorm*.

- Every uninorm *U* has a distinguished element *a* called *annihilator*, for which the following holds $U(a, x) = U(\mathbf{0}_L, \mathbf{1}_L) = a$. A uninorm *U* is said to be conjunctive if $U(x, \mathbf{0}_L) = \mathbf{0}_L$, and *U* is said to be disjunctive if $U(\mathbf{1}_L, x) = \mathbf{1}_L$, for all $x \in [\mathbf{0}_L, \mathbf{1}_L]$.

Lemma 1 (see [13]) Let *U* be a uninorm with the neutral element *e*. Then, for $(x, y) \in [0, 1]^2$ the following holds:

i. $T(x, y) = \frac{U(ex, ey)}{e}$ is a t-norm,

ii. $S(x, y) = \frac{U((1-e)x+e, (1-e)y+e)-e}{1-e}$ is a t-conorm.

For all $(x, y) \in A(e)$ we have

$$\min(x, y) \leqslant U(x, y) \leqslant \max(x, y).$$

The notion of locally internal operations on [0, 1] was introduced by Martín *et al.* [14]. This notion was generalized to operations (uninorms) on bounded lattices by Çayli *et al.* [15].

Definition 7 (see [14, 15]) Let *U* be a uninorm. We say that *U* is *locally internal* if $U(x, y) \in \{x, y\}$ for all $(x, y) \in [0, 1]^2$.

A uninorm *U* is *locally internal on a set* $G \subset [0, 1]^2$ if $U(x, y) \in \{x, y\}$ for all $(x, y) \in G$.

Let *L* be a bounded lattice. We say that a uninorm $U : L^2 \to L$ is *locally internal* if

$$U(x, y) \in \{x, y, x \wedge y, x \vee y\}.$$

Among locally internal uninorms, we will be interested mainly in those which are locally internal on a set *G*.

Remark 3.

(a) Particularly, a uninorm *U* (on [0, 1]) is *locally internal on the boundary* if $U(x, 0) \in \{x, 0\}$ and $U(x, 1) \in \{x, 1\}$ holds for all

$x \in [0, 1]$. Some examples of uninorms which are not locally internal on the boundary can be found, e.g., in Hliněná et al. [16,17,4], see also Figure 1.

(b)   An important family of uninorms is that of locally internal ones. Results of Drewniak and Drygaś [18] imply that, on the unit interval, the family of all locally internal uninorms is identical with that of idempotent uninorms. Some further study of locally internal uninorms can be found, e.g., in Drygaś [19] and in literature referenced therein.

Example 2. Now, we present the construction of the uninorm U_1 sketched in Figure 1. This construction has been published in Hliněná et al. [4]. We will analyze this uninorm in the last subsection. We choose function $f(z) = \frac{z}{2} - \frac{1}{8}$ that will represent values $U_1(\frac{1}{8}, z)$ for $z \in [\frac{1}{4}, \frac{3}{4}]$. Using this function and a representable uninorm U_r we compute all other values in the rectangle $]0, \frac{1}{4}[\times]\frac{1}{4}, \frac{3}{4}[$, and by commutativity we get also the values in the rectangle $]\frac{1}{4}, \frac{3}{4}[\times]0, \frac{1}{4}[$. In general, $f : [\frac{1}{4}, \frac{3}{4}] \to [0, \frac{1}{4}]$ is a function that is continuous, strictly increasing and fulfilling $f(\frac{1}{4}) = 0$, $f(\frac{1}{2}) = \frac{1}{8}$ and $f(\frac{3}{4}) = \frac{1}{4}$. These properties (and the way of the construction) guarantee that for arbitrary $x \in]0, \frac{1}{4}[$ and $z \in]0, \frac{1}{4}[$ there exists a unique $y \in]\frac{1}{4}, \frac{3}{4}[$ such that $U_1(x, y) = z$.

Let us now construct the values of U_1 in the rectangle $A =]0, \frac{1}{4}[\times]\frac{1}{4}, \frac{3}{4}[$. Using function f, for arbitrary $x \in]0, \frac{1}{4}[$ there exists unique $\bar{y} \in]\frac{1}{4}, \frac{3}{4}[$ such that $x = U(\frac{1}{8}, \bar{y})$. Namely, $\bar{y} = f^{-1}(x) = 2x + \frac{1}{4}$. For arbitrary $(x, y) \in A$ we get

$$U_1(x, y) = U_1\left(U_1\left(\frac{1}{8}, \bar{y}\right), y\right) = U_1\left(\frac{1}{8}, U_1(\bar{y}, y)\right), \quad (4)$$

and $U_1(\bar{y}, y) = U_r(\bar{y}, y)$. Since $U_1(\frac{1}{8}, z) = f(z)$ for $z \in [\frac{1}{4}, \frac{3}{4}]$, this implies $U_1(x, y) = U_1\left(\frac{1}{8}, U_r(\bar{y}, y)\right) = f(U_r(\bar{y}, y))$. Finally, using the definition of f and the formula for $\bar{y}$, we have that

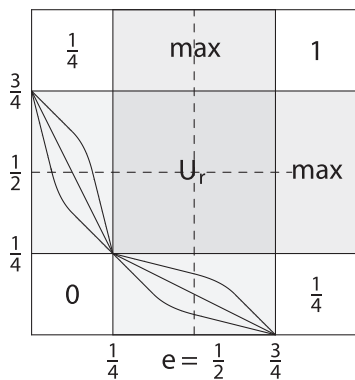$$U_1(x, y) = \frac{U_r(2x + \frac{1}{4}, y)}{2} - \frac{1}{8}. \quad (5)$$

Figure 1 | Uninorm U_1, example of a uninorm not locally internal on the boundary.

For $(x, y) \notin]0, \frac{1}{4}[\times]\frac{1}{4}, \frac{3}{4}[\cup]\frac{1}{4}, \frac{3}{4}[\times]0, \frac{1}{4}[$, we define

$$U_1(x, y) = \begin{cases} 0, & \text{if } \min\{x, y\} = 0, \\ & \quad \text{or if } \max\{x, y\} \leqslant \frac{1}{4}, \\ 1, & \text{if } \min\{x, y\} \geqslant \frac{3}{4}, \\ \frac{1}{4}, & \text{if } 0 < \min\{x, y\} \leqslant \frac{1}{4} \\ & \quad \text{and } \max\{x, y\} \geqslant \frac{3}{4}, \\ & \quad \text{or if } \min\{x, y\} = \frac{1}{4} \\ & \quad \text{and } \max\{x, y\} > \frac{1}{4}, \\ U_r(x, y), & \text{if } (x, y) \in]\frac{1}{4}, \frac{3}{4}[^2, \\ \max\{x, y\}, & \text{if } \frac{1}{4} < \min\{x, y\} < \frac{3}{4} \\ & \quad \text{and } \max\{x, y\} \geqslant \frac{3}{4}. \end{cases} \quad (6)$$

In Figure 1 we have sketched-level functions of U_1 for levels $\frac{1}{16}, \frac{1}{8}, \frac{3}{16}$ in the rectangles $]0, \frac{1}{4}[\times]\frac{1}{4}, \frac{3}{4}[$ and $]\frac{1}{4}, \frac{3}{4}[\times]0, \frac{1}{4}[$.

Concerning other types of uninorm on the unit interval we provide the following results by Drewniak and Drygaś [18], Martín et al. [14] and by Ruiz-Aguilera et al. [20].

Lemma 2 Let U be a uninorm. U is idempotent if and only if U is locally internal.

Proposition 3 (see, [21]) Let $f : [-\infty, \infty] \to [0, 1]$ be an increasing bijection. Then

$$U(x, y) = f^{-1}\left(f(x) + f(y)\right) \quad (7)$$

is a uninorm that is continuous everywhere except at points $(0, 1)$ and $(1, 0)$, and is strictly increasing on $]0, 1[^2$. U is conjunctive if we adopt the convention $-\infty + \infty = -\infty$, and U is disjunctive adopting the convention $-\infty + \infty = \infty$.

Definition 8 (see, [21]) The uninorm U fulfilling formula (7) for an increasing bijection $f : [-\infty, \infty] \to [0, 1]$ adopting either of the conventions, $-\infty + \infty = -\infty$ or $-\infty + \infty = \infty$, is said to be a *representable uninorm*.

Remark 4. Representable uninorms, under the name *aggregative operators* were studied already by Dombi [22].

Uninorms on bounded lattices with similar properties like the representable ones, have been constructed by Bodjanova and Kalina [23]. These uninorms utilize the notion of a commutative ℓ-group.

Another important class of uninorms is that of continuous ones on $]0, 1[^2$. These uninorms were characterized by Hu and Li [24], and further studied by Drygaś [25]. From results in Hu and Li [24] we have the following characterization.

Proposition 4 A uninorm U with neutral element $e \in]0, 1[$ is continuous on $]0, 1[^2$ if and only if one of the following conditions is satisfied:

i. *U is representable,*

ii. *there exists an element a with $0 < a < e$, a continuous t-norm T and representable uninorm U_r and an increasing bijection φ : $[a, 1] \to [0, 1]$ such that*
$U(x, y) = \varphi^{-1}\left(U_r(\varphi(x), \varphi(y))\right)$ *for* $(x, y) \in [a, 1]^2$,
$U(x, y) = aT(\frac{x}{a}, \frac{y}{a})$ *for* $(x, y) \in [0, a]^2$,
$U(x, y) = \min\{x, y\}$ *for* $(x, y) \in [0, a[\cap]a, 1[\cup]a, 1[\cap [0, a[$,
and U is locally internal on the boundary,

iii. *or there exists an element b with $e < b < 1$ a continuous t-conorm S and a representable uninorm U_r and an increasing bijection φ : $[0, b] \to [0, 1]$ such that*
$U(x, y) = \varphi^{-1}\left(U_r(\varphi(x), \varphi(y))\right)$ *for* $(x, y) \in [0, b]^2$,
$U(x, y) = b + (1 - b)S(\frac{x-b}{1-b}, \frac{y-b}{1-b})$ *for* $(x, y) \in [b, 1]^2$,
$U(x, y) = \max\{x, y\}$ *for* $(x, y) \in]b, 1] \cap [0.b[\cup [0, b[\cap]b, 1]$,
and U is locally internal on the boundary.

Some other important classes of uninorms were studied, e.g., in [26,27]. Now, we provide an overview of some families of uninorms on bounded lattices. Bodjanova and Kalina [23] have defined uninorms based on commutative lattice-ordered groups. Lattice-ordered groups were defined by Birkhoff [7].

Definition 9 (see [7]) Let $(\tilde{L}, \leqslant_{\tilde{L}})$ be a lattice and $(\tilde{L}, *)$ be a group such that for all x_1, x_2, y_1, y_2 fulfilling $x_1 \leqslant_{\tilde{L}} x_2$ and $y_1 \leqslant_{\tilde{L}} y_2$ the following holds

$$x_1 * y_1 \leqslant_{\tilde{L}} x_2 * y_2.$$

Then $(\tilde{L}, \leqslant_{\tilde{L}}, *)$ is said to be a *lattice ordered group*, ℓ-groups for brevity.

Proposition 5 (see [23]) *Let $(\tilde{L}, \leqslant_{\tilde{L}}, *)$ be a commutative ℓ-group. Set $L = \tilde{L} \cup \{\mathbf{0}_L, \mathbf{1}_L\}$ and organize $(L, \leqslant_L, \mathbf{0}_L, \mathbf{1}_L)$ into a bounded lattice with the bottom element $\mathbf{0}_L$ and the top element $\mathbf{1}_L$ that is an extension of $(\tilde{L}, \leqslant_{\tilde{L}})$. The function $U_c : L \times L \to L$ defined by*

$$U_c(x, y) = \begin{cases} x * y & \text{for } (x, y) \in \tilde{L}^2, \\ \mathbf{0}_L & \text{if } x = \mathbf{0}_L \text{ or } y = \mathbf{0}_L, \\ \mathbf{1}_L & \text{if } x = \mathbf{1}_L \text{ and } y \neq \mathbf{0}_L \\ & \text{or if } y = \mathbf{1}_L \text{ and } x \neq \mathbf{0}_L, \end{cases} \tag{8}$$

is a conjunctive uninorm. The function $U_d : L \times L \to L$ defined by

$$U_d(x, y) = \begin{cases} x * y & \text{for } (x, y) \in \tilde{L}^2, \\ \mathbf{1}_L & \text{if } x = \mathbf{1}_L \text{ or } y = \mathbf{1}_L, \\ \mathbf{0}_L & \text{if } x = \mathbf{0}_L \text{ and } y \neq \mathbf{1}_L \\ & \text{or if } y = \mathbf{0}_L \text{ and } x \neq \mathbf{1}_L, \end{cases} \tag{9}$$

is a disjunctive uninorm.

As Bodjanova and Kalina [23] noted, if L is a σ-complete lattice, we can define a limit

$$\lim_{n \to \infty} a_n = \begin{cases} \bigvee_{n=1}^{\infty} a_n & \text{for increasing sequences,} \\ \bigwedge_{n=1}^{\infty} a_n & \text{for decreasing sequences.} \end{cases}$$

Using the just defined limit the following assertion holds.

Lemma 6 (see [23]) Let L be a bounded lattice from Proposition 5 that is moreover σ-complete, and U_c be the conjunctive uninorm on L defined by (8). Then

$$\lim_{i \to \infty} U_c(a_i, b) = U_c\left(\lim_{i \to \infty} a_i, b\right),$$

where $(a_i)_{i=1}^{\infty}$ is a monotone sequence.

Remark 5.

1. An assertion similar to Lemma 6 could be formulated also the uninorm U_d from Proposition 5.

2. Lemma 6 shows that we have a kind of continuity for the uninorms U_c and U_d everywhere except of points $(\mathbf{0}_L, \mathbf{1}_L)$ and $(\mathbf{1}_L, \mathbf{0}_L)$. This means, these uninorms have properties similar to representable uninorms on $[0, 1]$.

Definition 10 The uninorms U_c and U_d defined in Proposition 5, will be called ℓ-*group-based uninorms.*

Birkhoff [7] introduced the notion of ordinal sum of bounded lattices. Let us have bounded lattices $(L_1, \leqslant_{L_1}, \mathbf{0}_{L_1}, \mathbf{1}_{L_1})$ and $(L_2, \leqslant_{L_2}, \mathbf{0}_{L_2}, \mathbf{1}_{L_2})$. We construct a new bounded lattice $(L, \leqslant_L, \mathbf{0}_{L_1}, \mathbf{1}_{L_2})$ in such a way that we "paste" the two lattices at elements $\mathbf{1}_{L_1}$ and $\mathbf{0}_{L_2}$. This means, we consider these two elements to be equal and the lattice order $\leqslant_L$ is given by

$$x \leqslant_L y \Leftrightarrow \begin{cases} x \in L_1 & y \in L_2, \\ x \leqslant_{L_1} y & \text{for } (x, y) \in L_1^2, \\ x \leqslant_{L_2} y & \text{for } (x, y) \in L_2^2. \end{cases}$$

Of course, we could also paste the lattices L_1 and L_2 by pasting them at $\mathbf{1}_{L_2}$ and $\mathbf{0}_{L_1}$. To distinguish these two possible ordinal sums, we will denote the former one by $(L_1 \cup L_2, \mathbf{0}_{L_1}, \mathbf{1}_{L_2})$ and the latter one by $(L_1 \cup L_2, \mathbf{0}_{L_2}, \mathbf{1}_{L_1})$.

For properties of ordinal sums of bounded lattices and the technic of pasting we recommend the paper by Riečanová [28].

Proposition 7 *Let $(\tilde{L}, \leqslant_{\tilde{L}}, *)$ be a commutative ℓ-group, $\mathbf{0}_L$ and $\mathbf{1}_L$ two distinguished elements and $(\hat{L}, \leqslant_{\hat{L}}, \mathbf{0}_{\hat{L}}, \mathbf{1}_{\hat{L}})$ a bounded lattice. Denote $L = \tilde{L} \cup \{\mathbf{0}_L, \mathbf{1}_L\}$ and let $(L, \leqslant_L, \mathbf{0}_L, \mathbf{1}_L)$ be the bounded lattice that extends $(\tilde{L}, \leqslant_{\tilde{L}})$ in the way as in Proposition 5. Further, denote $L_1 = (L \cup \hat{L}, \mathbf{0}_L, \mathbf{1}_{\hat{L}})$ and $L_2 = (L \cup \hat{L}, \mathbf{0}_{\hat{L}}, \mathbf{1}_L)$ the two possible ordinal sums of the lattices L and $\hat{L}$. Choose an ℓ-group-based uninorm $U_\ell : L \times L \to L$, a divisible t-norm $T : \hat{L} \times \hat{L} \to \hat{L}$ and a divisible t-conorm $S : \hat{L} \times \hat{L} \to \hat{L}$. Functions $U_1 : L_1 \times L_1 \to L_1$ and $U_2 : L_2 \times L_2 \to L_2$ fulfilling respectively*

$$U_1(x, y) = \begin{cases} U_\ell(x, y) & \text{for } (x.y) \in L^2, \\ T(x, y) & \text{for } (x, y) \in \hat{L}^2, \\ \min(x, y) & \text{if } \max(x, y) \in \tilde{L} \\ & \text{and } \min(x, y) < \mathbf{1}_{\hat{L}}, \\ \in \{x, y\} & \text{if } x = \mathbf{1}_L \text{ or } y = \mathbf{1}_L, \end{cases} \tag{10}$$

$$U_2(x,y) = \begin{cases} U_\ell(x,y) & for\ (x.y) \in L^2, \\ S(x,y) & for\ (x,y) \in \hat{L}^2, \\ \max(x,y) & if\ \max(x,y) > \mathbf{0}_{\check{L}} \\ & and\ \min(x,y) \in \tilde{L}, \\ \in \{x,y\} & if\ x = \mathbf{0}_L\ or\ y = \mathbf{0}_L, \end{cases} \quad (11)$$

are uninorms if and only if the partial functions $U_1(\mathbf{1}_L, \cdot) = U_1(\cdot, \mathbf{1}_L)$ and $U_2(\mathbf{0}_L, \cdot) = U_2(\cdot, \mathbf{0}_L)$ are monotone and there exists and idempotent element x_1 of T and x_2 of S such that

$$U_1(y, \mathbf{1}_L) = \begin{cases} \mathbf{1}_L & for\ y > x_1, \\ y & for\ y < x_1, \end{cases}$$

$$U_2(y, \mathbf{0}_L) = \begin{cases} \mathbf{0}_L & for\ y < x_2, \\ y & for\ y > x_2. \end{cases}$$

Proof. The construction of the uninorms is in fact an ordinal sum of a uninorm and a t-norm or a t-conorm, respectively. Hence we skip a detailed proof that the functions U_1 and U_2 are uninorms.

Definition 11 The uninorm U_1 constructed in Proposition 7 is said to be an ordinal sum of a divisible t-norm and an ℓ-group-based uninorm. The uninorm U_2 constructed in Proposition 7 is said to be an ordinal sum of an ℓ-group-based uninorm and a divisible t-conorm.

In Remark 3 we have pointed out the equality of the family of idempotent uninorms and that of locally internal uninorms on the unit interval. On bounded lattices the situation is different. It is straightforward that if a uninorm on a bounded lattice L is locally internal then it is idempotent. The next example shows that the converse implication is not true (more such examples can be found in Kalina [29]).

Example 3. Denote $L = [0,1]^2$ with the usual coordinate-wise ordering. Denote $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ and define

$$U(\mathbf{x}, \mathbf{y}) = \big(\max(x_1, y_1), \min(x_2, y_2)\big).$$

Then

$$U(\mathbf{0}_L, \mathbf{1}_L) = (1,0) \notin \{\mathbf{0}_L, \mathbf{1}_L, \mathbf{0}_L \wedge \mathbf{1}_L, \mathbf{0}_L \vee \mathbf{1}_L\},$$

hence U is idempotent but not locally internal.

2.2. An Overview of Pre-Orders Induced by a Semigroup

The study of orders (pre-orders) induced by a semigroup operation had started by Clifford [2]. Later, Hartwig [30] and independently also Nambooripad [31], defined a partial order on regular semigroups. Their definition is the following.

Definition 12 (see [30, 31]) Let $(S, \oplus)$ be a semigroup and E_S the set of its idempotent elements. Then

$$a \leqslant_\oplus b \ \Leftrightarrow \ (\exists e, f \in E_S)(a = b \oplus e = f \oplus b).$$

If the relation $\leqslant_\oplus$ is a partial order on S, it is called *natural*.

Definition 12 was generalized by Mitch [3].

Definition 13 (see [3]) Let $(S, \oplus)$ be an arbitrary semigroup. By $\lesssim_\oplus$ we denote the following relation

$$a \lesssim_\oplus b \ \Leftrightarrow \ a = b \oplus z_1 = z_2 \oplus b,\ a \oplus z_1 = a$$

for some $z_1, z_2 \in E_{S^1}$, where

$$S^1 = \begin{cases} S & if\ S\ has\ a\ neutral\ element, \\ S \cup \{e\} & otherwise, where\ e\ plays \\ & the\ role\ of\ the\ neutral\ element, \end{cases}$$

and E_{S^1} is the set of all idempotents of S^1.

Lemma 8 (see [3]) Let $(S, \oplus)$ be an arbitrary semigroup. The relation $\lesssim_\oplus$ is reflexive and anti-symmetric on S.

Proposition 9 (see [3]) *Let $(S, \oplus)$ be an arbitrary semigroup. The relation*

$$a \lesssim_\oplus b \ \Leftrightarrow \ a = x \oplus b = b \oplus y,\ a = x \oplus a \quad (12)$$

for some $x, y \in S^1$, is a partial order on S.

From now on, we restrict our attention to commutative semigroups. Lemma 8 and Proposition 9 immediately imply the following.

Lemma 10 Let $(S, \oplus)$ be a commutative semigroup. By $a_{\lesssim_\oplus}$ we denote the set

$$a_{\lesssim_\oplus} = \{z \in S : z \lesssim_\oplus a\},$$

where $a \in S$. Then for all $a, b \in S$ it holds that $a \lesssim_\oplus b$ if and only if $a_{\lesssim_\oplus} \subseteq b_{\lesssim_\oplus}$.

Directly by Definition 13 we get the following assertion.

Proposition 11 *Let $(S, \oplus)$ be a commutative semigroup. Then the set $a_{\lesssim_\oplus}$ is an ideal in $(S, \oplus)$.*

Lemma 12 Let $(S, \oplus)$ be a commutative semigroup. Let I_S be an ideal of $(S, \oplus)$. Then $(I_S, \oplus_{I_S})$ is a sub-semigrup of $(S, \oplus)$, where $\oplus_{I_S} = \oplus \upharpoonright I_S^2$.

Karaçal and Kesicioğlu [1] defined a partial order on bounded lattices L by means of t-norms.

Definition 14 (see [1]) Let L be a bounded lattice and $T : L \times L \to L$ a t-norm. We write $x \preceq_T y$ for arbitrary $x, y \in L$, if there exists $z \in L$ such that $x = T(y, z)$.

Proposition 13 (see [1]) *Let L be a bounded lattice and $T : L \times L \to L$ a t-norm. Then the relation $\preceq_T$ is a partial order on L.*

Remark 6. For arbitrary t-norm T, the partial order $\preceq_T$ from Definition 14 extends the partial order $\lesssim_T$ from Definition 13 in the following sense: let L be an arbitrary bounded lattice and T a commutative semigroup operation on L with a neutral element such that $(L, \preceq_T)$ is a partially ordered set. Then

$$a \lesssim_T b \ \Rightarrow \ a \preceq_T b$$

for all $a, b \in L$.

Important properties of the relation $\preceq_T$ by Karaçal and Kesicioğlu [1] are the following.

Proposition 14 (see [1]) *Let $T : L \times L \to L$ be a t-norm. Then*

(a) $\preceq_T \subset \leqslant_L$,

(b)   *T is divisible if and only if $\preceq_T = \leqslant_L$.*

Remark 7. Concerning a correspondence between properties of a binary aggregation function $A : L^2 \to L$ and the relation $\preceq_A$ (changing a t-norm T for A in Definition 14), the following can be said:

- if A has a neutral element, or A is idempotent, then $\preceq_A$ is reflexive,

- if A is associative, then $\preceq_A$ is transitive,

- the anti-symmetry of $\preceq_A$ fails if there exist elements $x \neq z$ and y_1, y_2 such that $z = A(x, y_1)$ and $x = A(z, y_2)$. Hence, if one of the following holds

$$(\forall (x, z) \in L^2) \quad x \preceq_A z \quad \Rightarrow \quad x \leqslant_L z,$$
$$(\forall (x, z) \in L^2) \quad x \preceq_A z \quad \Rightarrow \quad z \leqslant_L x,$$

then $\preceq_A$ is anti-symmetric. However, these two conditions are just sufficient as we may observe later in Example 5.

Hliněná *et al.* [4] introduced the following relation $\preceq_U$.

Definition 15 (see [4]) Let $U : [0, 1]^2 \to [0, 1]$ be an arbitrary uninorm. By $\preceq_U$ we denote the following relation

$$x \preceq_U y \text{ if there exists } \ell \in [0, 1] \text{ such that } U(y, \ell) = x.$$

Immediately by Definition 15 we get the next lemma.

Lemma 15 Let U be an arbitrary uninorm. Then $\preceq_U$ is transitive and reflexive. If a and e are the annihilator and the neutral elements of U, respectively, then

$$a \preceq_U x \preceq_U e$$

holds for all $x \in [0, 1]$.

Remark 8. In Definition 15 we have used the same notation $\preceq_U$ for the pre-order defined from a uninorm U, as in Definition 14 for the corresponding partial order $\preceq_T$ defined from a t-norm T. These two relations really coincide if $U = T$, i.e., the notation should not cause any problems.

The pre-order $\preceq_U$ extends the partial order $\lesssim_U$ from Proposition 9 in the following sense.

Proposition 16 *Let U be an arbitrary uninorm. Then*

$$x \lesssim_U y \quad \Rightarrow \quad x \preceq_U y \tag{13}$$

for all $(x, y) \in [0, 1]^2$.

A different type of partial order induced by uninorms has been defined by Ertuğrul *et al.* [5].

Definition 16 (see [5]) Let U be a uninorm and $e \in \,]0, 1[$ its neutral element. For $(x, y) \in [0, 1]^2$ denote $x \trianglelefteq_U y$ if one of the following properties is satisfied:

1.   there exists $\ell \in [0, e]$ such that $x = U(y, \ell)$ and $(x, y) \in [0, e]^2$,

2.   there exists $\ell \in [e, 1]$ such that $y = U(x, \ell)$ and $(x, y) \in [e, 1]^2$,

3. $0 \leqslant x \leqslant e \leqslant y \leqslant 1$.

Proposition 17 (see [5]) *For an arbitrary uninorm U, the relation $\trianglelefteq_U$ from Definition 16 is a partial order.*

Example 4. Consider the following uninorm U

$$U(x, y) = \begin{cases} \min(x, y) & \text{if } (x, y) \in [0, \tfrac{1}{2}]^2, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Then $\trianglelefteq_U$ coincides with the usual order of $[0, 1]$, while $x \lesssim_U y$ (see Proposition 9) if one of the following possibilities is satisfied

- $y \leqslant x$ for $x > 0.5$,

- $x \leqslant y$ for $(x, y) \in [0, 0.5[^2$,

- $y = 0.5$.

Remark 9. Let U be a uninorm. To compare the relation $\preceq_U$ from Definition 15 with $\trianglelefteq_U$ from Definition 16, the following should be remarked.

i.   The relation $\preceq_U$, given in Definition 15 is a pre-order, but not necessarily a partial order. Unlike this, the relation $\trianglelefteq_U$ defined by Definition 16, is always a partial order.

ii.   As illustrated by Example 4, the partial order $\trianglelefteq_U$ does not necessarily extends the partial order $\lesssim_U$ on the semigroup $([0, 1], U)$, i.e.,

$$x \lesssim_U y \Rightarrow \trianglelefteq_U y.$$

As shown by Proposition 16, the pre-order $\preceq_U$ always extends the partial order $\lesssim_U$ on $([0, 1], U)$, see formula (13).

Remark 10. Our intention is to characterize some families of uninorms by means of a (pre-)order they induce. As we have seen in this overview, we have at least three possibilities for the choice of an appropriate (pre-)order, namely that one by Mitsch [3] $\lesssim_U$ (Proposition 9), by Ertuğrul *et al.* citeEKK-16 $\trianglelefteq_U$ (Definition 16) and by Hliněná *et al.* [4] $\preceq_U$ (Definition 15). As we have pointed out in Proposition 16, the pre-order $\preceq_U$ extends $\lesssim_U$, this means the pre-order $\preceq_U$ has less incomparable pairs of elements of L then it is the case when using $\lesssim_U$ and thus, $\preceq_U$ can better characterize families of uninorms then $\lesssim_U$.

Concerning the partial order $\trianglelefteq_U$, it can well be used to characterize the underlying t-norm and t-conorm of a family of uninorms, however, it does not distinguish uninorms outside of $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$.

The above reasoning leads us to the choice of the pre-order $\preceq_U$ (Definition 15) to distinguish families of uninorms.

Definition 17 Let U be an arbitrary uninorm.

i.   For $(x, y) \in [0, 1]^2$ we denote $x \sim_U y$ if $x \preceq_U y$ and $y \preceq_U x$.

ii.   For $(x, y) \in [0, 1]^2$ we denote $x \parallel_U y$ if neither $x \preceq_U y$ nor $y \preceq_U x$ holds, and $x \nparallel_U y$ if $x \preceq_U y$ or $y \preceq_U x$.

iii.   For arbitrary $x \in [0, 1]$ we denote $x_{\sim_U} = \{z \in [0, 1] : z \sim_U x\}$.

3. SOME DISTINGUISHED FAMILIES OF UNINORMS AND PROPERTIES OF THE CORRESPONDING PREORDERS

We are going to study a relationship between some distinguished families $\mathcal{U}$ of uninorms on bounded lattices on the one hand and properties of the corresponding pre-orders $\preceq_U$ for $U \in \mathcal{U}$ on the other hand. When not otherwise stated, we will work with a bounded lattice $(L, \leqslant_L, \mathbf{0}_L, \mathbf{1}_L)$ (whose properties may be specified if necessary) and uninorms $U : L \times L \to L$.

As Deschrijver [32] has shown, except of conjunctive and disjunctive uninorms there exist also *uninorms of the third type*, namely those which are neither conjunctive nor disjunctive.

A direct consequence to Lemma 15 is the following.

Corollary 18 Let U be a uninorm. The following holds for all $x \in L$:

i. $\mathbf{0}_L \preceq_U x$ if and only if U is conjunctive,

ii. $\mathbf{1}_L \preceq_U x$ if and only if U is disjunctive,

iii. $a \preceq_U x$ where $a \notin \{\mathbf{0}_L, \mathbf{1}_L\}$ if and only if U is of the third type.

3.1. Locally Internal Uninorms

In this part we distinguish three types of locally internal uninorms:

1. on the boundary,

2. on $A(e)$,

3. on $[0, e]^2 \cup [e, 1]^2$.

Proposition 19 *Let U be a uninorm. It is locally internal on the boundary if and only if for every element $x \in L$*

$$\mathbf{0}_L \nparallel_U x \quad and \quad \mathbf{1}_L \nparallel_U x.$$

Proposition 20 *Let U be a uninorm with a neutral element e. Assume $I_e = \varnothing$. It is locally internal on $A(e)$ if and only if $\preceq_U$ is a partial order with the following properties:*

$$\left(\forall (x, y) \in [\mathbf{0}_L, e]^2\right) \quad (x \preceq_U y \Rightarrow x \leqslant_L y), \tag{14}$$

$$\left(\forall (x, y) \in [e, \mathbf{1}_L]^2\right) \quad (x \preceq_U y \Rightarrow y \leqslant_L x), \tag{15}$$

and $x \nparallel_U y$ for every $(x, y) \in A(e)$.

Proof. ($\Rightarrow$) If U is locally internal on $A(e)$ then $U(x, y) \in \{x, y\}$ for $(x, y) \in A(e)$. This and the fact that $I_e = \varnothing$ imply directly that $\preceq_U$ is a partial order with the required properties.

($\Leftarrow$) Let $(x, y) \in A(e)$ and $U(x, y) = z \notin \{x, y\}$. Without loss of generality we may assume $z < e$. Then $z \preceq_U x$ and at the same time $x \leqslant_L z$, which contradicts the constraint (14). This finishes the proof of the assertion.

Remark 11. For an arbitrary uninorm U and for a pair $(x, y) \in L^2$, we have

$$U(x, y) = x \quad \Rightarrow \quad x \preceq_U y,$$
$$U(x, y) = y \quad \Rightarrow \quad y \preceq_U x.$$

Results by Drygaś [19] imply that if a uninorm U is locally internal on $A(e)$, there are three possibilities:

(a) $U(x, y) = \min\{x, y\}$ for all $(x, y) \in A(e)$,

(b) $U(x, y) = \max\{x, y\}$ for all $(x, y) \in A(e)$,

(c) there exists a (not necessarily strictly) decreasing function $f : [0, e[\to]e, 1]$ such that, for $(x, y) \in [0, e[\times]e, 1]$, we have

$$U(x, y) \begin{cases} = x & \text{if } y < f(x), \\ = y & \text{if } y > f(x), \\ \in \{x, y\} & \text{if } y = f(x). \end{cases}$$

The next examples show what may happen if one of the constraints, (14) or (15), is not fulfilled, or if $I_e \neq \varnothing$, respectively.

Example 5. (See [33]) We present a uninorm $U_p : [0, 1]^2 \to [0, 1]$ with the neutral element $e = \frac{1}{2}$ constructed by "paving" (see Bodjanova and Kalina [33], Kalina and Král' [34] and Zhong et al. [35] for the construction technic). We have $I_0 =]0, \frac{1}{2}[, I_1 =]\frac{1}{2}, 1[$. $\varphi_1 :]0, \frac{1}{2}[\to]\frac{1}{2}, 1[$ is an increasing bijection, we can choose $\varphi_1(z) = z - \frac{1}{2}$. Further we set $T(x, y) = 2xy$ for $(x, y) \in [0, \frac{1}{2}]^2$, $\varphi_0]0, \frac{1}{2}[\to]0, \frac{1}{2}[$ to be the identity and $\varphi_2 :]0, \frac{1}{2}[\to \{1\}$. Then

$$U_p(x, y) = \varphi_{i+j}\left(T(\varphi_i^{-1}(x), \varphi_j^{-1}(j)\right)$$

for $x \in I_i$ and $y \in I_j$. Otherwise,

$$U_p(x, y) = \begin{cases} 0 & \text{if } \min(x, y) = 0, \\ 1 & \text{if } \max(x, y) = 1, \ \min(x, y) \neq 0, \\ x & \text{if } y = \frac{1}{2}, \\ y & \text{if } x = \frac{1}{2}. \end{cases}$$

In this case we get $x \preceq_{U_p} y$ if one of the following conditions is fulfilled:

$$x = 0$$
$$x = 1 \text{ and } y \neq 0$$
$$y = \frac{1}{2}$$
$$x \leqslant y \text{ for } (x, y) \in \left([0, 1] \setminus \{0, 1, \frac{1}{2}\}\right)^2.$$

The uninorm U_p, see Figure 2, is not locally internal on $A(e)$, though $\preceq_{U_p}$ is a linear order. Constrain (15) is violated.

Example 6. Now, we present a uninorm U with $I_e \neq \varnothing$ that is locally internal on $A(e)$ but $\preceq_U$ is not an order. Let $(\tilde{L}, \leqslant_{\tilde{L}}, *)$ be a commutative ℓ-group and $L_1 = \{\mathbf{0}_L, a, b, e, \mathbf{1}_L\}$. Organize $L = L_1 \cup \tilde{L}$ into a lattice which is ordered in the following way:

- $\mathbf{0}_L \leqslant_L a \leqslant_L e \leqslant_L b \leqslant_L \mathbf{1}_L$,

- $a \leqslant_L x \leqslant_L b$ for all $x \in \tilde{L}$,

- for $(x, y) \in \tilde{L}^2, x \leqslant_{\tilde{L}} y \Rightarrow x \leqslant_L y$.
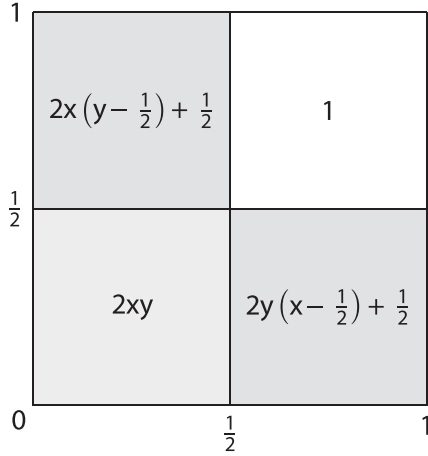
Figure 2 | The layout of the uninorm from Example 5.

On L we define the following uninorm:

$$
U(x,y) = \begin{cases}
x * y & \text{for } (x,y) \in \tilde{L}^2, \\
x \wedge_L y & \text{if } x \leqslant_L a \text{ or } y \leqslant_L a, \\
x \vee_L y & \text{for } (x,y) \in \{e,b,\mathbf{1}_L\}^2, \\
x & \text{if } y = e \\
& \text{or if } y \in \tilde{L}, x \geqslant_L b, \\
y & \text{if } x = e \\
& \text{or if } x \in \tilde{L}, y \geqslant_L b.
\end{cases}
$$

U is a uninorm with $I_e = \tilde{L}$ that is locally internal on $A(e)$ and $\preceq_U$ is not an order since $x \sim_U y$ for all $(x,y) \in \tilde{L}^2$.

Proposition 21 *Let U be a uninorm with $I_e = \varnothing$. U is locally internal on $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$ if and only if it is locally internal. In that case $\preceq_U$ is a linear order if and only if L is a chain.*

Proof. If a uninorm is locally internal on $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$ and with $I_e = \varnothing$, then it is idempotent. Hence, for $x \in [\mathbf{0}_L, e]$ and $y \in [e, \mathbf{1}_L]$

$$U(x,y) = U(U(x,x), y) = U(x, U(y,y)).$$

If $U(x,y) \in [\mathbf{0}_L, e]$ then

$$x \leqslant_L U(x,y) = U(x, U(x,y)) \leqslant_L x$$

and similarly for $U(x,y) \in [e, \mathbf{1}_L]$ we could prove $U(x,y) = y$. This implies that U is locally internal. Of course, local internality of U implies local internality on $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$. Assume that L is a chain. Then $\{x, y, x \wedge_L y, x \vee_L y\} = \{x, y\}$. This means that U is locally internal and hence $x \nparallel_U y$ for all $(x,y) \in L^2$. Thus, $\preceq_U$ is also a chain. On the other hand, assume there exist x, y such that $x \parallel_L y$. Let $(x,y) \in [\mathbf{0}_L, e]$. Then $U(x,y) = x \wedge_L y \notin \{x,y\}$ and thus, $x \parallel_U y$.

By the next example we illustrate what may happen if $I_e \neq \varnothing$.

Example 7. Assume $L = \{\mathbf{0}_L, \mathbf{1}_L, a, e\}$ is a bounded lattice with $a \parallel_L b$. The lattice L is a so-called diamond. The next tables define two uninorms on L with the neutral element e. Both uninorms are locally internal on $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$.

The uninorm U_1, see Table 1, generates the linear order $\preceq_{U_1}$

$$\mathbf{0}_L \preceq_{U_1} \mathbf{1}_L \preceq_{U_1} a \preceq_{U_1} e.$$

The uninorm U_2, see Table 2, generates the partial order $\preceq_{U_2}$

$$a \preceq_{U_2} \mathbf{0}_L \preceq_{U_2} e, \quad a \preceq_{U_2} \mathbf{1}_L \preceq_{U_2} e$$

and $\mathbf{0}_L \parallel_{U_2} \mathbf{1}_L$. Thus, when $I_e \neq \varnothing$, the local internality of a uninorm U on $[\mathbf{0}_L, e]^2 \cup [e, \mathbf{1}_L]^2$ is no guarantee that all elements of $[\mathbf{0}_L, e]$ are comparable with respect to $\preceq_U$ with all elements of $[e, \mathbf{1}_L]$.

3.2. Uninorms with Divisible Underlying T-Norm and T-Conorm

Results by Karaçal and Kesicioğlu [1] imply the following.

Proposition 22 *Let U be a proper uninorm with a neutral element e. Then U has divisible underlying t-norm and t-conorm if and only if the following holds:*

$$
\begin{aligned}
x \leqslant y &\Rightarrow x \preceq_U y & \text{for } (x,y) \in [\mathbf{0}_L, e]^2, \\
y \leqslant x &\Rightarrow x \preceq_U y & \text{for } (x,y) \in [e, \mathbf{1}_L]^2.
\end{aligned}
$$

The proof of Proposition 22 is omitted since it is a direct consequence of divisibility of t-norms and t-conorms. Recall that a special example of uninorms with divisible underlying t-norm and t-conorm are commutative ℓ-group-based uninorms where we have $x \sim_U y$ for all $(x,y) \in (L \setminus \{\mathbf{0}_L, \mathbf{1}_L\})^2$.

Propositions 20 and 22 have the following corollary.

Corollary 23 Let U be a proper uninorm with $I_e = \varnothing$. Then U is locally internal on $A(e)$ and with divisible underlying t-norm and t-conorm if and only if $x \nparallel_U y$ if and only if $x \nparallel_L y$.

Applying Proposition 7 to the pre-order $\preceq_U$ we get the following characterization of commutative ℓ-group-based uninorms.

Proposition 24 *A uninorm U is commutative ℓ-group-based if and only if for all $(x,y) \in (L \setminus \{\mathbf{0}_L, \mathbf{1}_L\})^2$ we have $x \sim_U y$.*

Proposition 7 implies the following characterization of the ordinal sum of a divisible t-norm or a divisible t-conorm and an ℓ-group-based uninorm (see Definition 11). The characterization is split into two propositions.

Table 1 | Uninorm U_1.

U_1	$\mathbf{0}_L$	a	e	$\mathbf{1}_L$
$\mathbf{0}_L$	$\mathbf{0}_L$	$\mathbf{0}_L$	$\mathbf{0}_L$	$\mathbf{0}_L$
a	$\mathbf{0}_L$	a	a	$\mathbf{1}_L$
e	$\mathbf{0}_L$	a	e	$\mathbf{1}_L$
$\mathbf{1}_L$	$\mathbf{0}_L$	$\mathbf{1}_L$	$\mathbf{1}_L$	$\mathbf{1}_L$

Table 2 | Uninorm U_2.

U_2	$\mathbf{0}_L$	a	e	$\mathbf{1}_L$
$\mathbf{0}_L$	$\mathbf{0}_L$	a	$\mathbf{0}_L$	a
a	a	a	a	a
e	$\mathbf{0}_L$	a	e	$\mathbf{1}_L$
$\mathbf{1}_L$	a	a	$\mathbf{1}_L$	$\mathbf{1}_L$

Proposition 25 *Let U be a proper uninorm with neutral element e. Then it is an ordinal sum of a divisible t-norm and an ℓ-group-based uninorm if and only if there exists $\mathbf{0}_L < a < e$ such that*

1. $x \sim_U y$ *for all* $(x, y) \in \,]a, \mathbf{1}_L[^2$,

2. $x \preceq_U y \Leftrightarrow x \leqslant y$ *for all* $(x,) \in [\mathbf{0}_L, a]^2$,

3. $I_a = \varnothing$.

Proof. The fact that the ordinal sum of a divisible t-norm and an ℓ-group-based uninorm induces a pre-order described in Proposition 25 is straightforward by Definition 11 and Proposition 7. We are going to prove the fact that if a pre-order fulfils the constraints of Proposition 25, it is induced by the ordinal sum of a divisible t-norm and an ℓ-group-based uninorm.

By Proposition 24 we have that $U \upharpoonright \,]a, \mathbf{1}_L[^2$ is a commutative ℓ-group operation.

Let $U(x, y) = z$ for $x \in [\mathbf{0}_L, a]$ and $y \in \,]a, \mathbf{1}_L[$. Because of monotonicity of U we have $z \leqslant_L a$. Of course, since $U \upharpoonright \,]a, \mathbf{1}_L[^2$ is a commutative ℓ-group operation, there exists y^{-1} such that $U(y, y^{-1}) = e$. We have three possibilities.

1. $z <_L x$. In this case $U(z, y^{-1}) = x$ and we have $x \preceq_U z$ which contradicts assumption 2 of the assertion in question.

2. $z >_L x$. This implies $z \preceq_U x$ and we have a contradiction with assumption 2. The above reasoning implies that only the third possibility, namely $z = x$ is not contradictory. Then we get by Proposition 14 that $U \upharpoonright [\mathbf{0}_L, a]^2$ is a divisible t-norm. This finishes the proof.

Proposition 26 *Let U be a proper uninorm with neutral element e. Then it is an ordinal sum of an ℓ-group-based uninorm and a divisible t-conorm if and only if there exists $e < b < \mathbf{1}_L$ such that*

1. $x \sim_U y$ *for all* $(x, y) \in \,]\mathbf{0}_L, b[^2$,

2. $x \preceq_U y \Leftrightarrow x \geqslant y$ *for all* $(x, y) \in [b, \mathbf{1}_L]^2$,

3. $I_b = \varnothing$.

We skip the proof of Proposition 26 since it follows the same idea as that of Proposition 25.

As a corollary to Propositions 25 and 26 we get the following.

Corollary 27 *Let $U : [0, 1]^2 \to [0, 1]$ be a proper uninorm with a neutral element e and different from a representable uninorm. Then U is continuous on $]0, 1[^2$ if and only if one of the following holds:*

I.   there exists $a < e$ such that
   1. $x \sim_U y$ for all $(x, y) \in \,]a, 1[^2$,

   2. $x \preceq_U y \Leftrightarrow x \leqslant y$ for all $(x, y) \in [0, a]^2$,

II.   there exists $b > e$ such that
   1. $x \sim_U y$ for all $(x, y) \in \,]0, b[^2$,

   2. $x \preceq_U y \Leftrightarrow x \geqslant y$ for all $(x, y) \in [b, 1]^2$.

3.3. Some Other Classes of Uninorms on *L* and on [0, 1]

First, we analyze the uninorm U_1 from Example 2. Looking at the layout of U_1 (Figure 1) we get pre-order that is induced by U_1. Particularly, the following holds:

Lemma 28 *Let U_1 be the uninorm from Example 2. Then*

- $x \sim_{U_1} y$ for all $(x, y) \in \,]0, \frac{1}{4}[^2$,

- $x \sim_{U_1} y$ for all $(x, y) \in \,]\frac{1}{4}, \frac{3}{4}[^2$,

- $0 \preceq_{U_1} \frac{1}{4} \preceq_{U_1} x$ for all $x \in \,]0, 1]$,

- $x \preceq_{U_1} y$ for all $x \in \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right]$ and $y \in \,]\frac{1}{4}, \frac{3}{4}[$,

- $1 \preceq_{U_1} x$ for all $x \in \,]\frac{1}{4}, 1]$,

- $x \parallel_{U_1} y$ for all $x \in \left[\frac{3}{4}, 1\right]$ and $y \in \,]0, \frac{1}{4}[$, and for $(x, y) \in \left[\frac{3}{4}, 1\right]^2$.

Now, we restrict our attention to $T_1 = U_1 \upharpoonright [0, \frac{1}{2}]^2$, this means, we restrict our attention to the underlying t-norm T_1. Considering the partial order $\preceq_{T_1}$ we get

Lemma 29 *Set $T_1 = U_1 \upharpoonright [0, \frac{1}{2}]^2$. Then*

- $x \preceq_{T_1} y$ if and only if $x \leqslant y$ for all $(x, y) \in \left[0, \frac{1}{4}\right]^2$,

- $x \preceq_{T_1} y$ if and only if $x \leqslant y$ for all $x \in [0, \frac{1}{2}]$ and $y \in \,]\frac{1}{4}, \frac{1}{2}]$,

- $x \parallel_{T_1} \frac{1}{4}$ for all $x \in \left[0, \frac{1}{4}\right[$,

and this implies that $\left(\frac{1}{4}, \frac{1}{4}\right)$ is the only discontinuity point of T_1.

Proof. Due to the construction of the uninorm U_1 in Example 2, for all $x \in \,]0, \frac{1}{4}[$ and all $y \in [0, x]$ there exists $z \in [\frac{1}{4}, 1]$ such that $U_1(x, z) = T_1(x, z) = y$. This implies that T_1 is continuous on $\left[0, \frac{1}{4}\right[\times [0, 1] \cup [0, 1] \times \left[0, \frac{1}{4}\right[$. Further, for all $x \in \,]\frac{1}{4}, \frac{1}{2}]$ and all $y \in [0, \frac{1}{2}]$ there exists $z \in [0, \frac{1}{2}]$ such that $T_1(x, z) = y$. Hence, T_1 is continuous also on $\left]\frac{1}{4}, \frac{1}{2}\right] \times [0, 1] \cup [0, 1] \times \left]\frac{1}{4}, \frac{1}{2}\right]$. Since $\frac{1}{4} \parallel_{T_1} x$ for all $x \in \,]0, \frac{1}{4}[$, we conclude that $(\frac{1}{4}, \frac{1}{4})$ is the only discontinuity point of T_1.

Next, we provide some results on uninorms with an area of constantness in $[0, e]^2$ or $[e, 1]^2$.

Proposition 30 (See [16]) *Let $U : [0, 1]^2 \to [0, 1]$ be a proper uninorm having e as neutral element. Let $y > e$ be an idempotent element of U. If there exists $x < e$ such that $U(x, y) = \tilde{x} \in \,]x, e[$ then*

$$U(z, y) = \tilde{x} \text{ and } U(z, x) = U(\tilde{x}, x) \quad \text{for all } z \in [x, \tilde{x}].$$

Proposition 30 can be directly generalized for uninorm on bounded lattices into the following form:

Proposition 31 *Let* $U : L \times L \to L$ *be a proper uninorm and* $y \notin \|_e$ *be an idempotent element. Assume there exists* $x \notin \|_e$ *such that* $U(x, y) = \tilde{x} \notin \{x, y\}$. *Then we have that*

1. *if* $x <_L \tilde{x} <_L e <_L y$ *then* $U(z, y) = \tilde{x}$ *and* $U(z, \tilde{x}) = U(z, x)$ *for all* $z \in [x, \tilde{x}]$,

2. *if* $x >_L \tilde{x} >_L e >_L y$ *then* $U(z, y) = \tilde{x}$ *and* $U(z, \tilde{x}) = U(z, x)$ *for all* $z \in [\tilde{x}, x]$.

Thus we get the following corollary.

Corollary 32 *Let* $U : L \times L \to L$ *be a proper uninorm having* e *as neutral element.*

i. *Assume* $y < e$ *is an idempotent element of* U. *Then either* $x \|_U y$ *for all* $x \in [e, 1]$ *or there exists an interval* $]a, b] \subset [e, 1]$ *such that* $y \|_U z$ *for all* $z \in]a, b]$.

ii. *Assume* $y > e$ *is an idempotent element of* U. *Then either* $x \|_U y$ *for all* $x \in [0, e]$ *or there exists an interval* $[a, b[\subset [0, e]$ *such that* $y \|_U z$ *for all* $z \in [a, b[$.

Kalina and Kráľ [34] introduced uninorms which are strictly increasing on $]0, 1[^2$, but not continuous. The construction method was further studied by Bodjanova and Kalina [33] and by Zong *et al.* [35]. Since we are not able to distinguish among continuous t-norms T (t-conorms S) by means of the relation $\preceq_T$ ($\preceq_S$), we are not able to characterize unambiguously uninorms which are strictly increasing on $]0, 1[^2$. We present the main idea of the construction method, paving, in case the basic "brick" is the product t-norm T_π:

(a) we split the interval $]0, 1[$ into infinitely countably many disjoint right-closed subintervals $\{I_j; j \in \mathcal{J}\}$, where $\mathcal{J}$ is an index set and $(\mathcal{J}, \circledast, j_0)$ is a commutative increasing monoid and j_0 is its neutral element,

(b) $\vartheta_j : I_j \to]0, 1]$ is an increasing bijection.

The resulting uninorm is defined by

$$U_p(x, y) = \vartheta_{i \circledast j}^{-1}\left(T_\pi(\vartheta_i(x), \vartheta_j(y))\right) \text{ for } x \in J_i, \ y \in J_j, \quad (16)$$

$$0 \quad \text{if } \min\{x, y\} = 0, 1 \quad \text{otherwise.}$$

Concerning the properties of $\preceq_{U_p}$ there are two possibilities depending whether $(\mathcal{J}, \circledast, j_0)$ is a group or not.

Proposition 33 *Let* U_p *be a uninorm defined by* (16), $(\mathcal{J}, \circledast, j_0)$ *be a commutative group and* $\{I_j; j \in \mathcal{J}\}$ *be a system of disjoint right-closed intervals whose union is* $]0, 1[$. *Then*

i. *for every* $j \in \mathcal{J}$ *and all* $(x, y) \in I_j^2$ *we have*

$$x \preceq_{U_p} y \Leftrightarrow x \leqslant y,$$

ii. *for all* $i, j \in \mathcal{J}$, $i \neq j$, *all* $x \in J_i$ *and* $y \in J_j$ *we have*

$$x \sim_{U_p} y \Leftrightarrow \vartheta_j(y) = \vartheta_i(x),$$
$$x \preceq l_{U_p} y \Leftrightarrow \vartheta_i(x) \leqslant \vartheta_j(y).$$

Proposition 34 *Let* U_p *be a uninorm defined by* (16), $(\mathcal{J}, \circledast, j_0)$ *be a commutative monoid without inverse elements, with the neutral element* j_0 *and* $\{I_j; j \in \mathcal{J}\}$ *be a system of disjoint right-closed intervals whose union is* $]0, 1[$. *Then*

i. *for every* $j \in \mathcal{J}$ *and all* $(x, y) \in I_j^2$ *we have*

$$x \preceq_{U_p} y \Leftrightarrow x \leqslant y,$$

ii. *for all* $i, j \in \mathcal{J}$, $i \neq j$, *all* $x \in J_i$ *and* $y \in J_j$ *we have* $x \preceq_{U_p} y \Leftrightarrow \vartheta_i(x) \leqslant \vartheta_j(y)$ *and* $(\exists k \in \mathcal{J})(j \circledast k = i)$, $x \|_{U_p} y$ *if and only if one of the following holds* $(\nexists k \in \mathcal{J})(j \circledast k = i \vee i \circledast k = j)$, $(\exists k \in \mathcal{J})(j \circledast k = i)$ *and* $\vartheta_i(x) > \vartheta_j(y)$, $(\exists k \in \mathcal{J})(i \circledast k = j)$ *and* $\vartheta_i(x) < \vartheta_j(y)$.

We could formulate dual theorems to Propositions 33 and 34 for the case when the basic "brick" is the probabilistic sum t-conorm.

4. CONCLUSIONS

In the paper we have reviewed known types of (pre-) orders induced by semigroups. Our main goal was to characterize some families of uninorms on the unit interval as well as on bounded lattices. We have chosen the pre-order introduced by Hliněná *et al.* [4] as the most appropriate for our intention. We have characterized uninorms which are locally internal on the boundary, on $A(e)$ and on $([\mathbf{0}_L, e] \cup [e, \mathbf{1}_L])^2$, uninorms with divisible uderlying operations, and some other types of uninorms. As a by-product, we have presented a t-norm with a single discontinuity point.

CONFLICTS OF INTEREST

We proclaim that there is no conflict of interest that could prevent the publication of the manuscript.

AUTHORS' CONTRIBUTIONS

Hereby I confirm, also from behalf of my co-author, that these are our original results and that the paper is an extension of the paper presented at the conference EUSFLAT 2019 in Prague.

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