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ON BASES OF CLOSURE OPERATORS ON COMPLETE LATTICES

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Abstract. We study closure operators on complete lattices and define for them the concepts of closed and open bases as well as that of a neighborhood base. Some properties of these concepts are studied including relationships between them. In particular, we find sufficient conditions under which the properties are analogous to those known for topological spaces.

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1. Introduction. Closure operators which are more general than the Kuratowski ones represent classical structures that have an important role across mathematics. In his pioneering paper [3] published as early as in 1937, E. Čech studied closure operators on a set X that are just the maps $u : \exp X \to \exp X$ (where $\exp X$ stands for the power set of X) which are grounded, extensive, and isotone. Considered in [3], too, the closure operators that are, moreover, idempotent are discussed in more detail by W. Sierpinski in [13], one of the first books to deal with the topic. They were also studied by many other authors – see, e.g., [14]. Idempotent closure operators are closely related to complete lattices, as already observed by G. Birkhoff in his classic book [2], and occur in many branches of mathematics. To give some examples, let us mention the transitive closure of a binary relation in set theory, the linear span of a set of vectors in linear algebra, the algebraic closure in algebra, the conjugate closure in group theory, and the convex hull in geometry. But the closure operators have numerous applications in other disciplines as well such as informatics (data analysis and knowledge representation – see [7]), formal logic (see [11]), physics (quantum mechanics – see [1] and [12]), etc. In these applications, the closure operators employed are even non-grounded in many cases.

The set theoretic closure operators may be generalized by being considered on partially ordered sets (or posets for short) rather than on the Boolean lattices $\exp X$. The generalized closure operators which are extensive, isotone, and idempotent are commonly used in lattice theory where, e.g., Galois connections give rise to them. They play an extremely important role in categorical topology when closure operators are studied on categories – see [5]. A closure operator on a category $\mathcal{X}$ is obtained when, for every object X of $\mathcal{X}$, a closure operator is given on the subobject lattice of X such that the morphisms $f : X \to Y$ in $\mathcal{X}$ are continuous with respect to the closure operators on the subobject lattices of X and Y. Thus, to investigate the behavior of a categorical closure operator means to study the properties of closure operators on the subobject lattices in the category under consideration – cf. [5]. Subobject lattices are always assumed to be complete lattices. In this paper, we study closure operators on general complete lattices so that the results achieved may be applied to the study of categorical closure operators. In particular, we focus on open, closed and neighborhood bases with respect to a closure operator on a complete lattice. We are particularly interested in finding conditions under which the bases behave analogously to the bases in topological spaces – cf. [6]. Although the subobject lattices in categories need not be complemented, in many cases, they are at least pseudocomplemented. Therefore, we will employ pseudocomplements instead of complements in complete lattices when studying closure operators on them. In [15], closure operators on certain posets are studied so that the present paper may be viewed as a continuation of [15].

2. Preliminaries. For the lattice-theoretic concepts used see, e.g., [9]. Recall that a lattice $X = (X, \leq)$ with a smallest element (denoted by 0) is said to be *atomistic* provided that every element of X is the join of a set of atoms of X. Clearly, every atomistic lattice is *atomic*, which means that, for every element $x \in X$ with $x \neq 0$, there is an atom $a \in X$ such that $a \leq x$. The *pseudocomplement* of an element $x \in X$ is the greatest element $y \in X$ with the property $x \wedge y = 0$ (provided it exists). In other words, y is the pseudocomplement of x, if, for every $z \in X$, we have $z \leq y \Leftrightarrow x \wedge z = 0$. The pseudocomplement of x will be denoted by $\overline{x}$. An element $x \in X$ is said to be *pseudocomplementable* if $\overline{x}$ exists and the lattice X is called *pseudocomplemented* if every element of X is pseudocomplementable. Of the properties of a pseudocomplemented lattice X, let us mention the following ones: For any $x, y \in X$ we have

(1) $x \leq \overline{\overline{x}}$,

(2) $x \leq y \Rightarrow \overline{y} \leq \overline{x}$,

(3) $\overline{x} = \overline{\overline{\overline{x}}}$.

If, in a pseudocomplemented lattice X, every element $x \in X$ satisfies $x = \overline{\overline{x}}$, then X is a Boolean lattice (with complements coinciding with pseudocomplements). We will also need the fact that, for a pseudocomplemented complete lattice X and a subset $A \subseteq X$, the equality $\overline{\bigvee A} = \bigwedge\{\overline{x};\ x \in A\}$ holds (while the equality $\overline{\bigwedge A} = \bigvee\{\overline{x};\ x \in A\}$ need not be true in general - here, only the inequality $\geq$ always holds). Let us also recall that an element x in a complete lattice X is called $\bigvee$-*prime* if, for any subset $A \subseteq X$, $x \leq \bigvee A$ implies that there is an element $y \in A$ such that $x \leq y$.

Example 2.1. Since pseudocomplements generalize the usual complements in lattices, every complemented lattice (particularly, every Boolean lattice) is pseudocomplemented. Pseudocomplemented lattices which are not complemented may often be found among lattices of subalgebras of a given algebra. A simple example of such a lattice is the lattice of subalgebras of the idempotent mono-unary algebra (X, f) with $X = \{0, 1\}$, $f(0) = 1$ and $f(1) = 1$. The three-element chain $\{\emptyset, \{1\}, \{0, 1\}\}$ is evidently a pseudocomplemented but not complemented complete lattice.

Note that every atom $p \in X$ has the property $p \le x$ or $p \le \overline{x}$ whenever $x \in X$ is a pseudocomplementable element (because $p \not\le x \Rightarrow p \wedge x = 0 \Rightarrow p \le \overline{x}$).

We will also need the concept of *adjoint maps*, also known as an *isotone Galois connection*, between posets. If X, Y are posets and $f : X \to Y$ and $g : Y \to X$ maps such that, for every $x \in X$ and every $y \in Y$, the equivalence $f(x) \le y \Leftrightarrow x \le g(y)$ holds, then f is called a *left adjoint* and g is called a *right adjoint*. The left and right adjoints f and g, respectively, between posets X and Y are isotone and determine each other because we have:

(1) $g(y) = \max\{x;\ f(x) \le y\}$ for every $y \in Y$ and

(2) $f(x) = \min\{y;\ g(y) \ge x\}$ for every $x \in X$.

If $f : X \to Y$ is a left adjoint between posets X and Y, we denote by $f^{\perp} : Y \to X$ the corresponding right adjoint. Note that we have $f(f^{\perp}(y)) \le y$ for every $y \in Y$ and $x \le f^{\perp}(f(x))$ for every $x \in X$. We will also work with the obvious facts that $f(0) = 0$, $f(\bigvee A) = \bigvee\{f(x);\ x \in A\}$ whenever $A \subseteq X$, and $f^{\perp}(\bigwedge B) = \bigwedge\{f^{\perp}(x);\ x \in B\}$ whenever $B \subseteq Y$.

Definition 2.2. Let $X = (X, \le)$ be a poset. A *closure operator* on X is a map $u : X \to X$ which fulfills the following three axioms:

(i) for all $x \in X$, $x \le u(x)$ (*extensiveness*),

(ii) for all $x, y \in X$, $x \le y \Rightarrow u(x) \le u(y)$ (*isotonicity*),

(iii) for all $x \in X$, $u(u(x)) = u(x)$ (*idempotency*).

If u us a closure operator on a poset X, then an element $x \in X$ is said to be *closed* (with respect to u) if $u(x) = x$.

Example 2.3. A classical example of closure operators on posets is obtained by composing a pair of antitone mappings that constitute an (antitone) Galois connection between a pair of posets. A number of further examples may be found in [5] where closure operators are considered on subobject lattices in various categories (to obtain closure operators on the categories).

In the sequel, we restrict our considerations to closure operators on complete lattices.

If u is a closure operator on a complete lattice X, then the pair (X, u) is called a *closure system*.

Remark 2.4. We distinguish between closure systems and closure spaces: by a *closure space* we understand, as usual, a pair (X, u) where X is a (generally non-ordered) set and u is a closure operator on X in the usual sense, i.e., a closure operator on the Boolean lattice $\exp X = (\exp X, \subseteq)$ in accordance with Definition 2.2.

A closure operator u on a complete lattice X and the closure system (X, u) are called

(iv) *grounded* if $u(0) = 0$,

(v) *additive* if $u(x \vee y) = u(x) \vee u(y)$ for all $x, y \in X$.

The well-known concept of a continuous map is transferred from the classical closure operators to our more general setting as follows:

DEFINITION 2.5. Let (X, u) and (Y, v) be closure systems. A left adjoint $f : X \to Y$ is said to be a *continuous map* from (X, u) into (Y, v) if $f(u(x)) \leq v(f(x))$ for every $x \in X$.

PROPOSITION 2.6. *Let (X, u) and (Y, v) be closure systems and $f : X \to Y$ be a left adjoint. Then, f is a continuous map of (X, u) into (Y, v) if and only if $u(f^{\perp}(y)) \leq f^{\perp}(v(y))$ for every $y \in Y$.*

Proof. Let f be continuous. Then, for every $y \in Y$, we have $f(u(f^{\perp}(y))) \leq v(f(f^{\perp}(y))) \leq v(y)$, hence $u(f^{\perp}(y)) \leq f^{\perp}(v(y))$. Conversely, let $u(f^{\perp}(y)) \leq f^{\perp}(v(y))$ for every $y \in Y$. Then, for every $x \in X$, we have $u(x) \leq u(f^{\perp}(f(x))) \leq f^{\perp}(v(fx))$, hence $f(u(x)) \leq v(f(x))$. Therefore, f is continuous. □

Let X and Y be lattices and $f : X \to Y$, $g : Y \to X$ maps. We say that f and g satisfy the *Frobenius reciprocity law* if, whenever $x \in X$ and $y \in Y$,

$$f(x \wedge g(y)) = f(x) \wedge y.$$

As usual, given posets X and Y with least elements, a map $f : X \to Y$ is said to *reflect* 0 if, for every $x \in X$, $f(x) = 0 \Leftrightarrow x = 0$. Clearly, if f is a left adjoint, then it reflects 0 if and only if $f^{\perp}(0) = 0$.

We will need the following observation:

LEMMA 2.7. *Let (X, u), (Y, v) be closure systems and let $f : (X, u) \to (Y, v)$ be a continuous map reflecting 0 such that f and $f^{\perp}$ satisfy the Frobenius reciprocity law. Then, for every pseudocomplementable element $y \in Y$, $f^{\perp}(y)$ is pseudocomplementable too with $\overline{f^{\perp}(y)} = f^{\perp}(\overline{y})$.*

Proof. Let $y \in Y$ be an element. Then, we have $f^{\perp}(\overline{y}) \wedge f^{\perp}(y) = f^{\perp}(\overline{y} \wedge y) = f^{\perp}(0) = 0$. Let $x \in X$ be an element with $x \wedge f^{\perp}(y) = 0$. Then, $f(x) \wedge y = f(x \wedge f^{\perp}(y)) = 0$. Hence, $f(x) \leq \overline{y}$ and, consequently, $x \leq f^{\perp}(f(x)) \leq f^{\perp}(\overline{y})$. Therefore, $f^{\perp}(\overline{y}) = \overline{f^{\perp}(y)}$. □

EXAMPLE 2.8. (1) In the case of a pair of closure spaces (see Remark 2.4), continuous maps between them are understood to be the maps between their underlying sets such that their liftings to the power sets are continuous maps between the corresponding closure systems. Of course, such liftings f are left adjoints between the power sets (with $f^{\perp} = f^{-1}$) reflecting 0 (i.e., empty set) and satisfying the Frobenius reciprocity law.

(2) A closure operator on a category $\mathcal{X}$ is always considered with respect to a given $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms in $\mathcal{X}$. In a category with a closure operator, for every morphism $f : A \to B$, a map between the subobject lattice of A and that of B is defined by assigning to a subobject m of A the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of $f \circ m$. These maps and their inverses (given by pullbacks) satisfy the Frobenius reciprocity law if (and only if), in the $(\mathcal{E}, \mathcal{M})$-factorization structure for morphisms, $\mathcal{E}$ is stable under pullbacks along $\mathcal{M}$-morphisms (cf. [4]).

We will extend some basic topological concepts (see, e.g., [6]) to closure systems and such extended concepts will be studied.

3. Closed, open, and neighborhood bases.

Definition 3.1. (Cf. [10], Definition 5.3(3)) Let (X, u) be a closure system. A pseudocomplementable element $x \in X$ is said to be *open* if $\overline{x}$ is closed, i.e., if $\overline{x} = u(\overline{x})$.

Remark 3.2. Let (X, u) be a closure system and $x \in X$ a pseudocomplementable element. Then, we clearly have:

(1) If $\overline{\overline{x}} = x$, then x is closed if and only if $\overline{x}$ is open;

(2) x is open if and only if $x \leq \overline{u(\overline{x})}$ (because $\overline{x} = u(\overline{x}) \Rightarrow x \leq \overline{\overline{x}} = \overline{u(\overline{x})}$ and, conversely, $x \leq \overline{u(\overline{x})} \Rightarrow \overline{x} \geq \overline{\overline{u(\overline{x})}} \geq u(\overline{x})$).

Proposition 3.3. *Let (X, u), (Y, v) be closure systems and let $f : (X, u) \to (Y, v)$ be a continuous map reflecting 0 such that f and $f^{\perp}$ satisfy the Frobenius reciprocity law. If $y \in Y$ is an open element, then $f^{\perp}(y)$ is open, too.*

Proof. Let $y \in Y$ be an open element. Then, $\overline{y}$ is closed and, therefore, $u(f^{\perp}(\overline{y})) \leq f^{\perp}(v(\overline{y})) = f^{\perp}(\overline{y})$ by Proposition 2.6. Thus, $f^{\perp}(\overline{y})$ is closed and, since $f^{\perp}(\overline{y}) = \overline{f^{\perp}(y)}$ by Lemma 2.7, $\overline{f^{\perp}(y)}$ is closed, too. Thus, $f^{\perp}(y)$ is open. □

Definition 3.4. Let u be a closure operator on a complete lattice X. A subset $A \subseteq X$ is called a *closed base* (*open base*) of u if every element of A is closed (open) and, for every closed (open) element $x \in X$, there is a subset $B \subseteq A$ such that $u(x) = \bigwedge B$ ($u(x) = \bigvee B$).

Example 3.5. 1) Given a complete lattice X, every subset $A \subseteq X$ is a closed base of a closure operator on X, namely the closure operator u given by $u(x) = \bigwedge\{y \in A;\ x \leq y\}$ for every $x \in X$.

2) Given a closure operator u on a complete lattice X, the set of all closed elements is a (largest) closed base of u.

3) Of course, for Kuratowski closure operators (i.e., topologies), the defined concepts of closed and open bases coincide with the usual ones.

The following proposition gives a necessary and sufficient condition for a given subset of a complete lattice to be a closed base of a given closure operator on the lattice.

PROPOSITION 3.6. *Let (X,u) be a closure system and let $A \subseteq X$ be a subset. Then, A is a closed base of u if and only if, for every pair of elements $x, y \in X$, there holds $y \le u(x) \Leftrightarrow \forall z \in A:\ x \le z \Rightarrow y \le z$.*

Proof. Let A be a closed base of u and let $x, y \in X$ be elements. Suppose that $y \le u(x)$ and let $z \in A$ be an element with $x \le z$. Then, $u(x) \le z$ because z is closed. Hence, $y \le z$. Conversely, let $x \le z \Rightarrow y \le z$ for every $z \in A$. Since $u(x)$ is closed, there exists a subset $B \subseteq A$ such that $u(x) = \bigwedge B$. Consequently, $x \le \bigwedge B \le t$ for every $t \in B$. Thus, $y \le t$ for every $t \in B$. Therefore, $y \le \bigwedge B = u(x)$. We have proved the equivalence $y \le u(x) \Leftrightarrow \forall z \in A:\ x \le z \Rightarrow y \le z$.

For every pair of elements $x, y \in X$, let us have $y \le u(x) \Leftrightarrow \forall z \in A:\ x \le z \Rightarrow y \le z$. Let v be a closure operator on X such that A is a closed base of v and let $x \in X$ be a closed element with respect to v. Then, there is a subset $B \subseteq A$ such that $x = \bigwedge B$. Since $u(x) \le u(x)$ and $x \le y$ for every $y \in B$, we have $u(x) \le y$ for every $y \in B$. Therefore, $u(x) \le \bigwedge B = x$. We have shown that x is closed with respect to u. Conversely, let $x \in X$ be an element closed with respect to u. Since $v(x) = \bigwedge\{y \in A;\ x \le y\}$, we have $x \le y \Rightarrow v(x) \le y$ for every $y \in A$. Thus, $v(x) \le u(x) = x$ and we have shown that x is closed with respect to u. Hence, closed sets with respect to u coincide with those with respect to v. Therefore, $u = v$. □

REMARK 3.7. For a closure system (X,u), put $\rho_u = \{(x,y) \in X^2;\ x \le u(y)\}$. It is shown in [15] that the assignment $u \mapsto \rho_u$ is a bijection between the set of all closure operators on X and the set of all preorders ρ on X satisfying the following two conditions:

(1) For all $x, y \in X$, $x \le y \Rightarrow x\rho y$.

(2) If $x_i \in X$ for every $i \in I$ ($I \ne \emptyset$ a set) and $x \in X$, then $\bigvee\{x_i;\ i \in I\}\rho x$ whenever $x_i \rho x$ for every $i \in I$.

Thus, in Proposition 3.6, we may write $y\rho_u x$ instead of $y \le u(x)$.

PROPOSITION 3.8. *Let (X,u) be a closure system such that X is pseudocomplemented and $y = \overline{\overline{y}}$ is valid for every closed element $y \in X$. If $A \subseteq X$ is an open base of u, then $\{\overline{x};\ x \in A\}$ is a closed base of u.*

Proof. Since A is an open base of u, $\overline{x}$ is closed for every $x \in A$. Let $y \in X$ be a closed element, i.e., let $y = u(y)$. Then, $\overline{\overline{y}} = u(\overline{\overline{y}})$, so that $\overline{y}$ is open. Therefore, there is a subset $B \subseteq A$ such that $\overline{y} = \bigvee B$. Hence, $y = \overline{\overline{y}} = \overline{\bigvee B} = \bigwedge\{\overline{x};\ x \in B\}$. Since $\{\overline{x};\ x \in B\} \subseteq \{\overline{x};\ x \in A\}$, $\{\overline{x};\ x \in A\}$ is a closed base of u. □

REMARK 3.9. Unfortunately, Proposition 3.8 is not valid when interchanging "closed" and "open". But, of course, if (X,u) is a closure system such that X is a Boolean lattice and $A \subseteq X$ is a closed base of u, then $\{\overline{x};\ x \in A\}$ is an open base of u.

In the following definition, the usual concept of neighborhoods of a subset of a topological space is extended to closure operators on complete lattices.

Definition 3.10. (Cf. [8], Definition 3.1.) Let (X, u) be a closure system and $x \in X$ an element. A pseudocomplementable element $y \in X$ is called a *neighborhood* of x if $x \wedge u(\overline{y}) = 0$. We denote by $\mathcal{N}_u(x)$ (or briefly $\mathcal{N}(x)$) the set of all neighborhoods of x. A subset $\mathcal{B} \subseteq \mathcal{N}_u(x)$ is called a *neighborhood base* at x if, for every $y \in \mathcal{N}_u(x)$, there is $z \in \mathcal{B}$ such that $z \leq y$.

Thus, for any pair of elements $x, y \in X$ such that y and $u(y)$ are pseudocomplementable, we have $y \in \mathcal{N}_u(x)$ if and only if $x \leq \overline{u(\overline{y})}$. Some basic properties of neighborhoods are listed below:

Proposition 3.11. *Let (X, u) be a closure system and $x, y \in X$ elements. Then*

(1) $1 \in \mathcal{N}(x)$ *if u is grounded (here, 1 denotes the greatest element of X),*

(2) $\mathcal{N}(0) = \{y \in X;\ y$ *pseudocomplementable*$\}$,

(3) *if $x > 0$, then $y > 0$ for every $y \in \mathcal{N}(x)$,*

(4) *$y \in \mathcal{N}(x)$ implies $x \leq y$ provided that (a) x is an atom or (b) $\overline{\overline{y}} = y$ and $u(\overline{y})$ is pseudocomplementable,*

(5) *if $y \in \mathcal{N}(x)$ and $z \in X$ is pseudocomplementable element with $z \geq y$, then $z \in \mathcal{N}(x)$,*

(6) $x \leq y \Rightarrow \mathcal{N}(y) \subseteq \mathcal{N}(x)$,

(7) *if $x > 0$ and $y_1, y_2, ..., y_k \in \mathcal{N}(x)$ $(k \in \mathbb{N})$, then $x \wedge y_1 \wedge y_2 \wedge ... \wedge y_k > 0$,*

(8) *if $y_1, y_2 \in \mathcal{N}(x)$, then $y_1 \wedge y_2 \in \mathcal{N}(x)$ provided that u is additive and X is a Boolean lattice,*

(9) *$x \in \mathcal{N}(x)$ if and only if x is open,*

(10) *if y is open and $x \leq y$, then $y \in \mathcal{N}(x)$.*

Proof. To prove (4), let $y \in \mathcal{N}(x)$. (a) Suppose that x is an atom. Then, $x \wedge u(\overline{y}) = 0$, which yields $x \not\leq u(\overline{y})$. Therefore, $x \not\leq \overline{y}$, so that $x \leq y$. (b) If $\overline{\overline{y}} = y$ and $u(\overline{y})$ is pseudocomplementable, then $x \wedge u(\overline{y}) = 0$ and $\overline{y} \leq u(\overline{y})$ imply $x \leq \overline{u(\overline{y})} \leq \overline{\overline{y}} = y$.

To prove (7), let $x > 0$, $y \in \mathcal{N}(x)$, and suppose that $x \wedge y = 0$. Then, $x \wedge u(\overline{y}) = 0$ and $x \leq \overline{y} \leq u(\overline{y})$. Thus, we have $x = x \wedge u(\overline{y}) = 0$, which is a contradiction. Hence, (7) is valid for $k = 1$. Suppose that it is valid for some $k \in \mathbb{N}$. Let $x > 0$ and $y_1, y_2, ..., y_k, y_{k+1} \in \mathcal{N}(x)$. Then, $x \wedge y_1 \wedge y_2 \wedge ... \wedge y_k > 0$ and, by (6), $y_{k+1} \in \mathcal{N}(x \wedge y_1 \wedge y_2 \wedge ... \wedge y_k)$, thus $x \wedge y_1 \wedge y_2 \wedge ... \wedge y_k \wedge y_{k+1} > 0$ (because (7) is valid for $k = 1$). This proves (7). The remaining assertions are obvious. □

Proposition 3.12. *Let* (X,u) *and* (Y,v) *be closure systems and let* $f:(X,u)\to(Y,v)$ *be a continuous map. Then, for all* $x\in X$ *and* $y\in Y$, $y\in\mathcal{N}(f(x))$ *implies* $f^{\perp}(y)\in\mathcal{N}(x)$.

Proof. Let $x\in X$, $y\in\mathcal{N}(f(x))$, and assume that $f^{\perp}(y)\notin\mathcal{N}(x)$. Since $f^{\perp}(y)$ is pseudocomplementable (with $f^{\perp}(\overline{y})=\overline{f^{\perp}(y)}$ by Lemma 2.7), we have $x\wedge u(\overline{f^{\perp}(y)})>0$. Therefore, $f(x)\wedge f(u(f^{\perp}(\overline{y})))\geq f(x\wedge u(\overline{f^{\perp}(y)}))>0$. Since $f(u(f^{\perp}(\overline{y})))\leq v(f(f^{\perp}(\overline{y})))\leq v(\overline{y})$, we have $f(x)\wedge v(\overline{y})>0$. This is a contradiction. □

Proposition 3.13. *Let* (X,u) *be a closure system, let* $x,z\in X$ *be elements,* $x>0$, *and let* $B\subseteq\mathcal{N}(x)$ *be a neighborhood base at* x. *If* $x\leq u(z)$, *then* $y\wedge z>0$ *for every* $y\in B$. *The converse is true if* X *is pseudocomplementable,* x *is an atom of* X, *and* $\overline{\overline{z}}=z$.

Proof. Let $x\leq u(z)$ and let $y\in B$ be an element with $y\wedge z=0$. Then, $z\leq\overline{y}$, hence $u(z)\leq u(\overline{y})$. Thus, since $x\wedge u(\overline{y})=0$, we have $x\wedge u(z)=0$. But this is a contradiction with $0<x\leq u(z)$.

Conversely, let $y\wedge z>0$ for every $y\in B$. Of course, then $y\wedge z>0$ for every $y\in\mathcal{N}(x)$. Let X be pseudocomplemented and let x be an atom of X, and let $\overline{\overline{z}}=z$. Suppose that $x\not\leq u(z)$. Then, $x\leq\overline{u(z)}=\overline{u(\overline{\overline{z}})}$, thus $x\wedge u(\overline{\overline{z}})=0$. Therefore, $\overline{z}\in\mathcal{N}(x)$. Consequently, there exists $t\in B$, $t\leq\overline{z}$ and, since $\overline{z}\wedge z=0$, we have $t\wedge z=0$. This is a contradiction. Therefore, $x\leq u(z)$. □

Corollary 3.14. *Let* (X,u) *be a closure system with* X *pseudocomplemented and let* $z\in X$ *be an element with* $\overline{\overline{z}}=z$. *If* $u(z)$ *equals the join of atoms of* X *and* $B_x\subseteq\mathcal{N}(x)$ *is a neighborhood base at* x *for every atom* $x\in X$, *then* $u(z)=\bigvee\{x\in X;\ x$ *is an atom of* X *and* $y\wedge z>0$ *for all* $y\in B_x\}$.

Corollary 3.15. *Let* X *be an atomistic Boolean lattice and* u,v *be closure operators on* X. *If, for every atom* $x\in X$, *there are neighborhood bases* $B_1\subseteq\mathcal{N}_u(x)$ *and* $B_2\subseteq\mathcal{N}_v(x)$ *at* x *such that* $B_2\subseteq B_1$, *then* $u\leq v$.

Theorem 3.16. *Let* (X,u) *be a closure system with* X *atomistic and* $B\subseteq X$ *be a set of open elements. If the set* $B_a=\{x\in B;\ a\leq x\}$ *is a neighborhood base at* a *for every atom* $a\in X$, *then* B *is an open base of* u.

Proof. Let B_a be a neighborhood base at a for every atom $a\in X$ and let $y\in X$ be an open element. Let T be the set of all atoms of X and put $T_y=\{a\in T;\ a\leq y\}$. Then, for every $a\in T_y$, we have $y\in\mathcal{N}(a)$ (see Proposition 3.11(10)). Therefore, there is an element $x_a\in B_a$ with $x_a\leq y$. Then, $\{x_a;\ a\in T_y\}\subseteq\bigcup_{a\in T}B_a=B$ (the last equality follows from the fact that X is atomic) and $\bigvee\{x_a;\ a\in T_y\}\leq y$. Since X is atomistic, we also have $y=\bigvee T_y$. Further, $x_a\in B_a$ implies $x_a\in\mathcal{N}(a)$, hence $a\leq x_a$ for every $a\in T_y$ by Proposition 3.11(4). Consequently, $y=\bigvee T_y\leq\bigvee\{x_a;\ a\in T_y\}$. Therefore, $y=\bigvee\{x_a;\ a\in T_y\}$. This completes the proof. □

LEMMA 3.17. *Let (X, u) be a closure system and $x, y \in X$ be elements such that $\overline{\overline{y}} = y$ and $\overline{\overline{u(\overline{y})}} = u(\overline{y})$. Then, $y \in \mathcal{N}(x)$ if and only if there is an open element $z \in \mathcal{N}(x)$ such that $z \leq y$.*

Proof. Let $y \in \mathcal{N}(x)$ and put $z = \overline{u(\overline{y})}$. Then, $z \leq y$ (because $\overline{y} \leq u(\overline{y})$ implies $y = \overline{\overline{y}} \geq \overline{u(\overline{y})} = z$) and, since $x \wedge u(\overline{y}) = 0$ and $u(\overline{y}) = u(u(\overline{y})) = u(\overline{\overline{u(\overline{y})}})$, we have $x \wedge u(\overline{\overline{u(\overline{y})}}) = 0$. Thus, $z \in \mathcal{N}(x)$ and $u(\overline{\overline{u(\overline{y})}}) = u(\overline{y}) \leq \overline{\overline{u(\overline{y})}}$, so that z is open. Conversely, if there is an open element $z \in \mathcal{N}(x)$ such that $z \leq y$, then clearly $y \in \mathcal{N}(x)$ (see Proposition 3.11(5)). □

THEOREM 3.18. *Let (X, u) be a closure system and $x \in X$ be a $\bigvee$-prime element such that $\overline{\overline{t}} = t$ and $\overline{\overline{u(\overline{t})}} = u(\overline{t})$ for every $t \in \mathcal{N}(x)$. If $A \subseteq X$ is an open base on u, then the set $B = \{y \in A;\ x \leq y\}$ is a neighborhood base at x.*

Proof. Let $A \subseteq X$ be an open base of u. Clearly, every $y \in B$ is a neighborhood of x because $x \wedge u(\overline{y}) \leq y \wedge u(\overline{y}) = y \wedge \overline{y} = 0$. Let $t \in \mathcal{N}(x)$. By Lemma 3.17, there is an open element $z \in \mathcal{N}(x)$ such that $z \leq t$. Since z is open, there is a subset $C \subseteq A$ such that $z = \bigvee C$. Then $x \leq z$ and, since x is $\bigvee$-prime, there exists $y \in C$ such that $x \leq y$. Then $y \leq z$, hence $y \leq t$, and $y \in B$. The proof is complete. □

COROLLARY 3.19. *Let (X, u) be a closure system with X a Boolean lattice and $A \subseteq X$ be an open base of u. Then the set $B = \{y \in A;\ x \leq y\}$ is a neighborhood base at x for every $\bigvee$-prime element $x \in X$.*

4. Conclusion. In this note, several well known results on bases in topological spaces are generalized to closure systems. Namely, atomistic Boolean complete lattices are, up to isomorphisms, power sets ordered by set inclusion (cf. [9]). Therefore, for closure spaces, i.e., closure systems of the form $(\exp X, u)$ where X is a set, the hypotheses of each of the assertions presented in the paper are satisfied. In the particular case of u being a Kuratowski closure operator, we receive well known facts on the behavior of bases in topological spaces.

In the literature, many topological concepts and results can be found extended to a categorical level, i.e., to categories with closure operators - see, e.g., [5]. While neighborhoods and neighborhood bases with respect to a categorical closure operator were introduced and studied in [8], the concepts of closed and open bases have not yet been considered for such an operator. These concepts, which are introduced and studied for closure operators on complete lattices in this paper, may naturally be transferred to closure operators on a category $\mathcal{X}$ because such closure operators are simply families $(u_X)_{X \in \mathcal{X}}$ where, for every object $X \in \mathcal{X}$, u_X is a closure operator on the subobject lattice of X. Of course, all the above results proved for closed, open, and neighborhood bases of closure operators on complete lattices will hold for categorical closure operators as well (though, in that case, the lattices may be large, becoming classes).

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