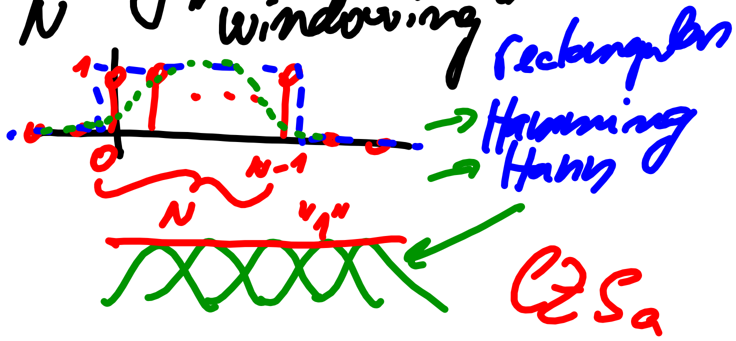
Operations with discrete signals.

1. Limitation to length N "windowing"

$y[n] = R_N[n]\,x[n]$



2. Periodization

$z[n] = y[\mathrm{mod}_N(n)]$

3. Periodization with a shift

n	-3	-2	-1	0	1	2	3	4	5	6	7	8
$y[n]$				4	3	2	1					
$\mathrm{mod}_4(n)$	1	2	3	0	1	2	3	0	1	2	3	
$\mathrm{mod}_4(n-2)$	0	1	2	3	0	1	2	3	0	1	2	3
$y[\mathrm{mod}_4(n-2)]$	4	3	2	1	4	3	2	1	4	3	2	1

CONVOLUTION

$x_1[n] = [1\ 2\ 3\ 4]$

$x_2[n] = [1\ 1\ -1\ -1]$

1. Linear convolution $y[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x[k]\,x_2[n-k]$

$\sum_{k=-\infty}^{\infty} x_1[n-k]\,x_2[k]$

n	0	1	2	3	4	5	6	7	8	...
$y[n]$	1	3	4	4	-1	-7	-4	0	0	

$2N-1$ samples 2x sequence of length $N \rightarrow 2N-1$

Multiplication of polynomials!

$(x^3+2x^2+3x+4)(x^3+x^2-x-1) =$ Try it out!

2. Periodic (cyclic) convolution

$y[n] = x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]\,x_2[\mathrm{mod}_N(n-k)]$

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	
$x_1[k]$				1	2	3	4						
$x_2[\mathrm{mod}_N(n-k)]$	-1	-1	1	1	-1	-1	1	1	-1	-1	1	1	...
$y[n]$	... 0	-4	0	4	0	-4	0	4	0	-4	0	4	0 -4 0 4

output with infinite length, periodic every 4 samples.

3. circular convolution – I want only N samples!

$y[n] = x_1[n] \, \textcircled{N} \, x_2[n] = R_N[n] \sum_{k=0}^{N-1} x_1[k]\,x_2[\mathrm{mod}_N(n-k)]$

output has N samples

discrete time

DFT $x_1[n] \rightarrow X_1[k]$

$x_2[n] \rightarrow X_2[k]$

$x_1[n] \, \textcircled{N} \, x_2[n] \longrightarrow X_1[k]X_2[k]$

continuous time:

$x_1(t) \rightarrow X_1(j\omega)$

$x_2(t) \rightarrow X_2(j\omega)$

$x_1(t) * x_2(t) \longrightarrow X_1(j\omega)X_2(j\omega)$

CZS a

Frequency analysis of discrete signals.

n	0	1	2	3
$x[n]$	2	2	0	0

Discrete-time Fourier transform DTFT

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

periodicity of spectrum 2π

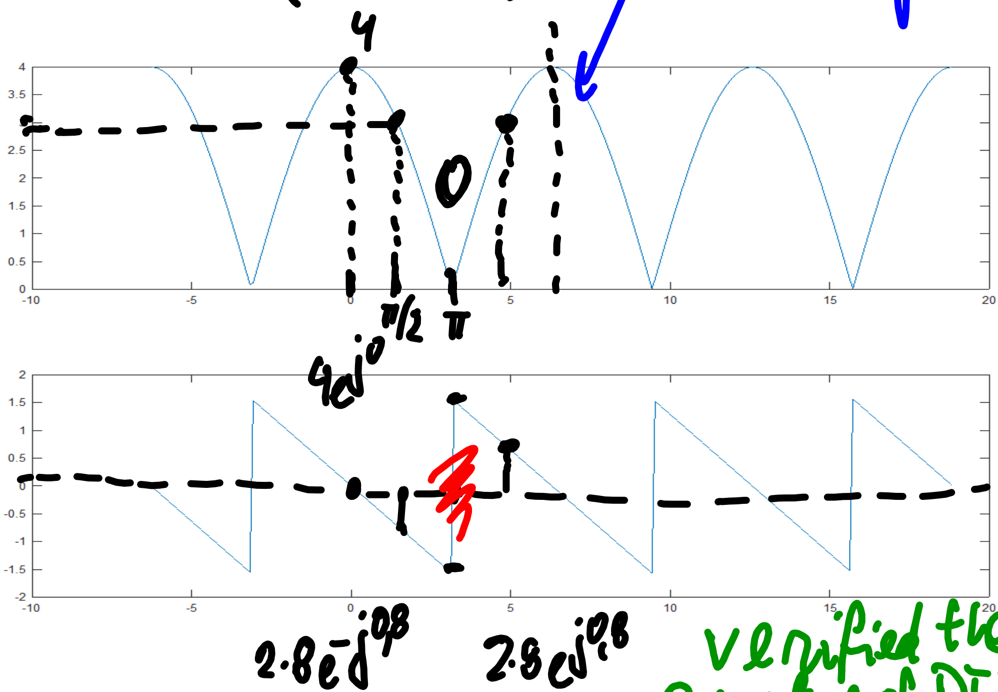
normalized angular frequency

$$\tilde{X}(e^{j\omega}) = x[0]\underbrace{e^{-j\omega 0}}_{1} + x[1]e^{-j\omega 1} = 2 + 2e^{-j\omega}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tilde{X}(e^{j\omega}) = \tilde{X}^{*}\left(e^{j(2\pi - \omega)}\right)$$

continuous spectrum



verified the sampling of DTFT!

Freq. analysis of a periodic discrete signal

n	0	1	2	3	
$x[n]$	2	2	0	0	
$\tilde{x}[n]$ 2 2 0 0	2	2	0	0	2 2 0 0 ...

$\tilde{x}[n] = x[\mathrm{mod}_4(n)]$

period of N samples

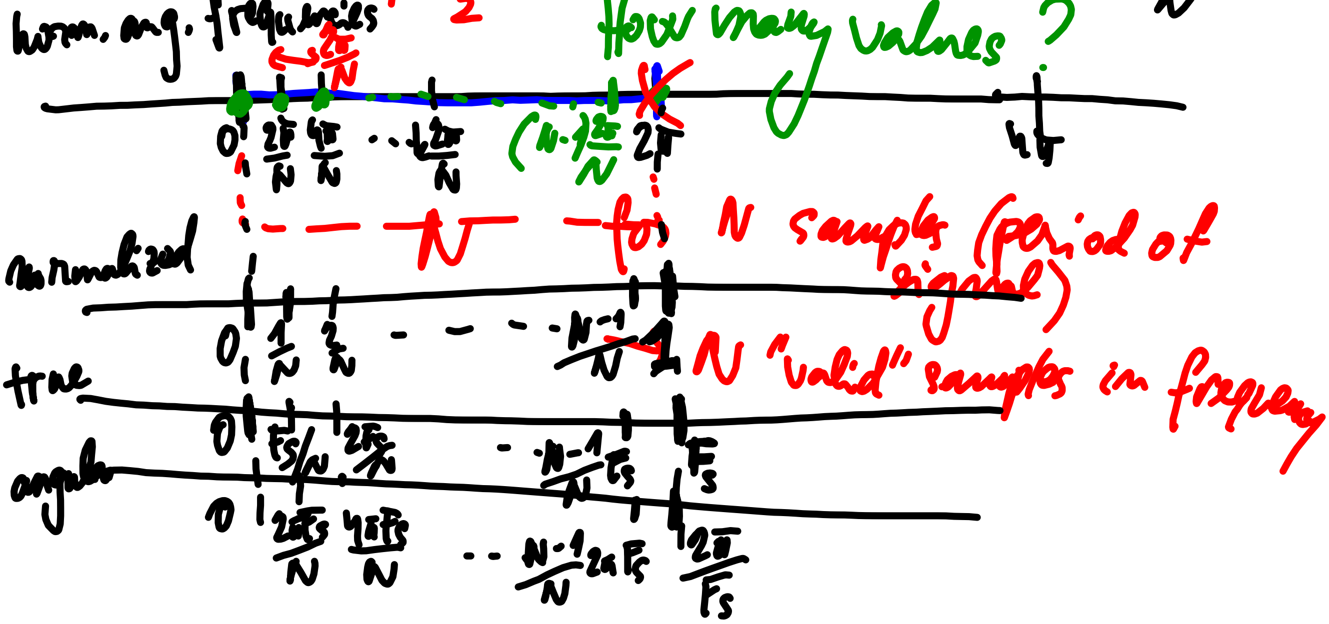
time	frequency
sampling	periodization (every 2π ...)
periodization	Sampling (discretization)

History: continuous time:
$x(t)$ with period T_1, $\omega_1 = \frac{2\pi}{T_1}$ → coefficients of F. series c_k "sitting" on $k\omega_1$

Discrete signals:
$\tilde{x}[n]$ with period of N samples $\omega_1 = \frac{2\pi}{N}$ → coefficients of Discrete Fourier Series sitting at $k\omega_1 = k\frac{2\pi}{N}$

fundamental angular frequency.

On example $\omega_1 = \frac{2\pi}{4} = \frac{\pi}{2}$



Discrete Fourier series – sampling DTFT!

DTFT: $\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$ any norm. ang. frequency

DFS: $\boxed{\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n}}$ multiples of fundamental freq.

Properties: 1. Periodicity is N coefficients

Proof: $\tilde{X}[k+mN] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j(k+mN)\frac{2\pi}{N}n} =$

$e^{a+b} = e^a \cdot e^b$

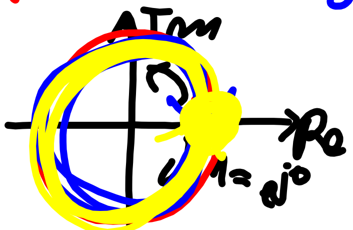
$= \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n} \cdot \cancel{e^{-jmN\frac{2\pi}{N}n}}$

$\cancel{e^{-jmn2\pi}}$

m - integer
n - integer
mn - integer.

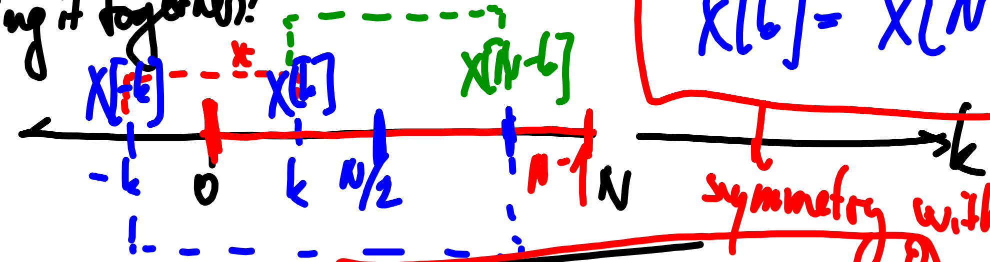
$= \tilde{X}[k]$ ∎

$e^{-jmn2\pi} = 1$



Symmetry: $\tilde{X}[-k] = \tilde{X}^*[k]$

Putting it together:



$\boxed{\tilde{X}[k] = \tilde{X}^*[N-k]}$ symmetry within interval $0 \ldots N-1$

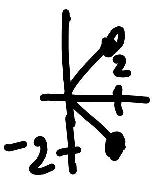
Inverse DFS: $\boxed{\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k]\, e^{+jk\frac{2\pi}{N}n}}$

Compute DFS of $\underbrace{2\ 2\ 0\ 0}_{\text{period } N=4},\ 2\ 2\ 0\ 0\ \ldots$

$$\tilde{X}[k] = \sum_{n=0}^{3} x[n]\, e^{-jk\frac{2\pi}{N}n} \qquad e^{-jk\frac{2\pi}{4}n} = e^{-jk\frac{\pi}{2}n}$$

		0	1	2	3	$\tilde{X}[k]$	
$x[n]$		2	2	0	0		
$k=0$	$e^{-j0}=1$	1	1	1	1	4	$4e^{j0}$
$k=1$	$e^{-j1\frac{\pi}{2}n} = e^{-j\frac{n\pi}{2}}$	1	$-j$	-1	j	$2-2j$	$2{,}8\,e^{-j\frac{\pi}{4}}$
$k=2$	$e^{-j\frac{2\pi}{2}n} = e^{-jn\pi}$	1	-1	1	-1	0	0
$k=3$	$e^{-j\frac{3\pi}{2}n}$	1	j	-1	$-j$	$2+2j$	$2{,}8\,e^{j\frac{\pi}{4}}$

$$\tilde{X}[1] = \tilde{X}^*[4-1] = \tilde{X}^*[3]$$



Q: is it really sampling the DTFT of a non-periodic signal $[2\ 2\ 0\ 0]$ at multiples of $\frac{2\pi}{4} = \frac{\pi}{2}$???

Discrete Fourier transform – DFT

"the ultimate tool of IT guys"

signal of N samples ($\mathbb{R}$) $\xrightarrow{DFT}$ spectrum of N samples

In theory:

$$\underset{0 \dots N-1}{x[n]} \xrightarrow[\text{(N samples)}]{\text{periodization}} \tilde{x}[n] \xrightarrow{DFS} \tilde{X}[k] \xrightarrow[\text{out N samples}]{\text{cutting}} X[k] = R_N[k]\tilde{X}[k]$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$ run only for $k = 0 \dots N-1$

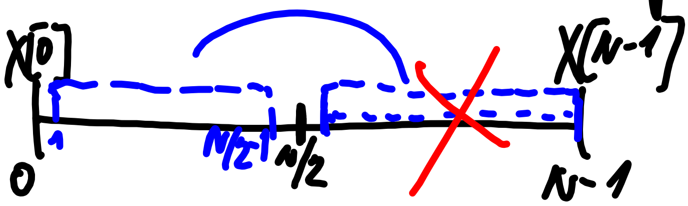
DFT: $$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}}$$ for $k \in [0, N-1]$

IDFT: $$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j\frac{2\pi}{N}kn}}$$

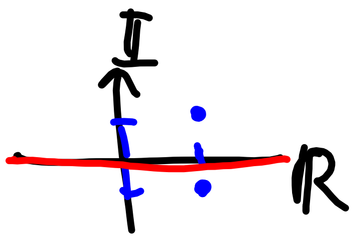
Symmetry within $k = 0 \dots N-1$

$X[N-k] = X^*[k]$

Counting no. of real numbers in the output.



$X[0] = \sum_{n=0}^{N-1} x[n] e^{0} = \sum x[n]$ 1 $\mathbb{R}$



$X[\frac{N}{2}] = X^*[N - \frac{N}{2}] = X^*[\frac{N}{2}]$ 1 $\mathbb{R}$

$X[1] \dots X[\frac{N}{2}-1] = X^*[N-1] \dots X^*[\frac{N}{2}+1]$ $2(\frac{N}{2}-1)$ $\mathbb{R}$

$1 + 1 + 2(\frac{N}{2}-1) = \underline{\underline{N}}$ real numbers again

DFT of 1 period of cosine

$x[n] = C_1 \cos\left(\frac{2\pi}{N}n + \varphi_1\right) = \frac{C_1}{2} e^{j\varphi_1} e^{j\frac{2\pi}{N}n} + \frac{C_1}{2} e^{-j\varphi_1} e^{-j\frac{2\pi}{N}n}$

$\cos x = \frac{e^{jx} + e^{-jx}}{2}$

$x[n] = \frac{1}{N} \sum X[k] e^{+j\frac{2\pi}{N}kn}$

$\frac{C_1}{2} e^{-j\varphi_1} e^{j\left(2\pi - \frac{2\pi}{N}\right)n} =$

$= \frac{C_1}{2} e^{-j\varphi_1} e^{j(N-1)\frac{2\pi}{N}n}$

$k = 1$

$X[1] = \frac{N C_1}{2} e^{j\varphi_1}$

$k = N-1$

$X[N-1] = \frac{N C_1}{2} e^{-j\varphi_1}$

abs. value of $\frac{NC_1}{2}$

phase $+\varphi_1$, $-\varphi_1$

just $X[1]$ and $X[N-1]$ non-zero

History: Fourier series: $x(t) = C_1 \cos(\omega_1 t + \varphi_1)$

$c_1 = \frac{C_1}{2} e^{j\varphi_1}$

$c_{-1} = C_1/2\, e^{-j\varphi_1}$

DFT vs. circular convolution

$$x_1[n] \longrightarrow X_1[k]$$

$$x_2[n] \longrightarrow X_2[k]$$

$$x_1[n] \, Ⓝ \, x_2[n] \longrightarrow X_1[k]X_2[k]$$

n	0	1	2	3
x_1	2	2	0	0
x_2	1	−1	0	0
Ⓝ	2	0	−2	0

k	$X_1[k]$	$X_2[k]$	$X_1[k]X_2[k]$
0	4	0	0
1	2−2j	1+1j	4
2	0	2	0
3	2+2j	1−1j	4

DFT $\longrightarrow$ FFT fast Fourier transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$ pre-computed

all coeffs!

N^2 complex multiplications

N^2 complex addition

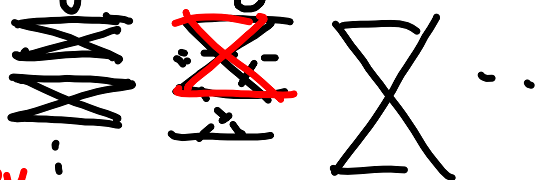
$N \log_2 N$

only powers of 2 : $N = 2^b$

stage 0 stage 1 . . . stage b-1

butterfly algo

Cooley-Tuckey



$\boxed{C \varepsilon S_a}$

How to apply DFT to compute "analog" things:

FS

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)\, e^{-jk\omega_1 t}\, dt$$

FT

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

DFT/FFT