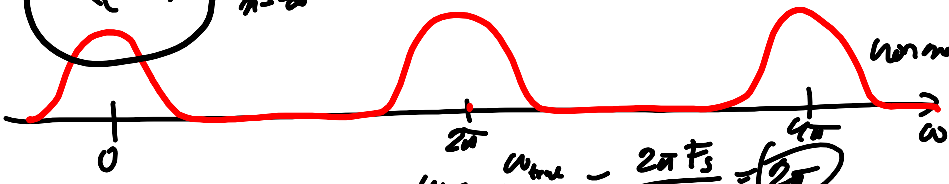
time	frequency
periodization	sampling / discretization - only coefficients "sitting" at the multiples of fundamental freq.
sampling	periodization - spectrum repeats at each multiple of the sampling freq.

Spec. analysis of discrete signals - DTFT

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

ω ← normalized angular frequency [rad]



$$\omega = \frac{\omega_{true}}{F_s} = \frac{2\pi F_s}{F_s} = 2\pi$$

Spec analysis of periodic discrete signals.

n	-3	-2	-1	0	1	2	3	4	5	6	
$\tilde{x}[n]$	2	0	0	2	2	0	0	2	2	0	0

$N = 4$

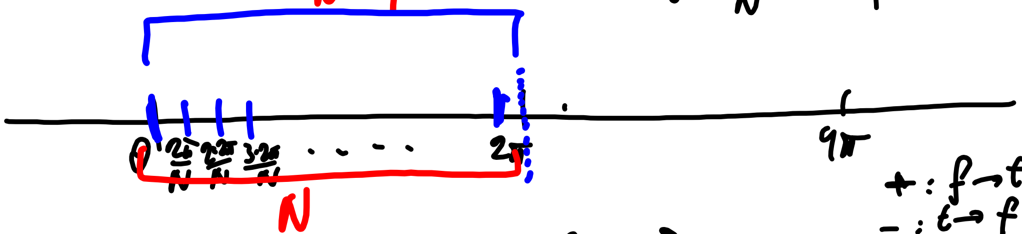
$f = \frac{1}{N}$

$\omega = \frac{2\pi}{N}$

Remember $x(t)$ T_1

$f_1 = \frac{1}{T_1}$ [Hz]

$\omega_1 = \frac{2\pi}{T_1}$ [rad/s]



+ : f → t (synthesis)

− : t → f (analysis)

Discrete Fourier Series (DFS)

output = magic constant, sum op., input, $e^{\pm j}$ (+ time, − freq.)

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n}$$

sitting at frequency $k\frac{2\pi}{N}$

period N samples

Inverse DFS

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k]\, e^{+jk\frac{2\pi}{N}n}$$

Proof, that DFS is periodic with N coefficients in frequency

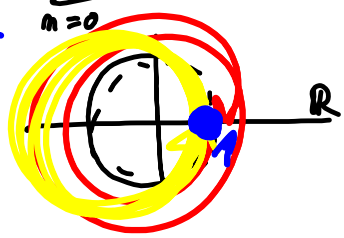
$\tilde{X}[k+m_1N]$ = need to prove that it is $\tilde{X}[k]$

$$\tilde{X}[k+m_1N] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j(k+m_1N)\frac{2\pi}{N}n} = \sum \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n}\cancel{e^{-jm_1N\frac{2\pi}{N}n}} =$$

$e^{a+b} = e^a e^b$

$e^{-jm_1 n 2\pi} = 1$

integer · integer = integer

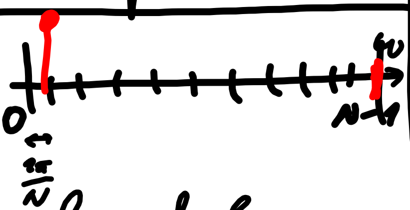


$$= \sum \tilde{x}[n]\, e^{-jk\frac{2\pi}{N}n} = \tilde{X}[k]$$

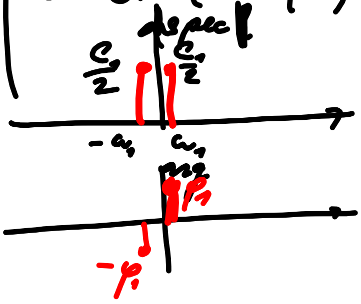
it is periodic with N

DFS of cosine with period N

$x[n] = C_1 \cos\left(\frac{2\pi}{N} n + \varphi_1\right)$



Remember
$x(t) = C_1 \cos(\omega_1 t + \varphi_1)$



lazy way: write synthesis formula and play Sherlock Holmes ...

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{+jk\frac{2\pi}{N}n}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$x[n] = \frac{C_1}{2} e^{j\varphi_1} e^{j\frac{2\pi}{N}n} + \frac{C_1}{2} e^{-j\varphi_1} e^{-j\frac{2\pi}{N}n} \longrightarrow \frac{C_1}{2} e^{-j\varphi_1} e^{j2\pi n} e^{-j\frac{2\pi}{N}n} = \frac{C_1}{2} e^{-j\varphi_1} e^{j2\pi n\left(\frac{N-1}{N}\right)}$$

$$2\pi n - \frac{2\pi}{N} n = \frac{2\pi n N - 2\pi n}{N} = 2\pi n\left(\frac{N-1}{N}\right)$$

$k=1$: $\frac{1}{N}\tilde{X}[1] = \frac{C_1}{2} e^{j\varphi_1}$ $\qquad$ $k=-1$: $\frac{1}{N}\tilde{X}[-1] = \frac{C_1}{2} e^{-j\varphi_1}$

$$\tilde{X}[1] = \frac{NC_1}{2} e^{j\varphi_1} \qquad \tilde{X}[-1] = \frac{NC_1}{2} e^{-j\varphi_1} = \tilde{X}[N-1]$$

$\tilde{X}[1], \tilde{X}[N-1]$: magnitude $\frac{NC_1}{2}$

phase φ_1 for $\tilde{X}[1]$ and $-\varphi_1$ for $\tilde{X}[N-1]$

DFT: have N samples in time $x[n]$ for $n \in 0 \dots N-1$
want N samples in freq: $X[k]$ for $k \in 0 \dots N-1$

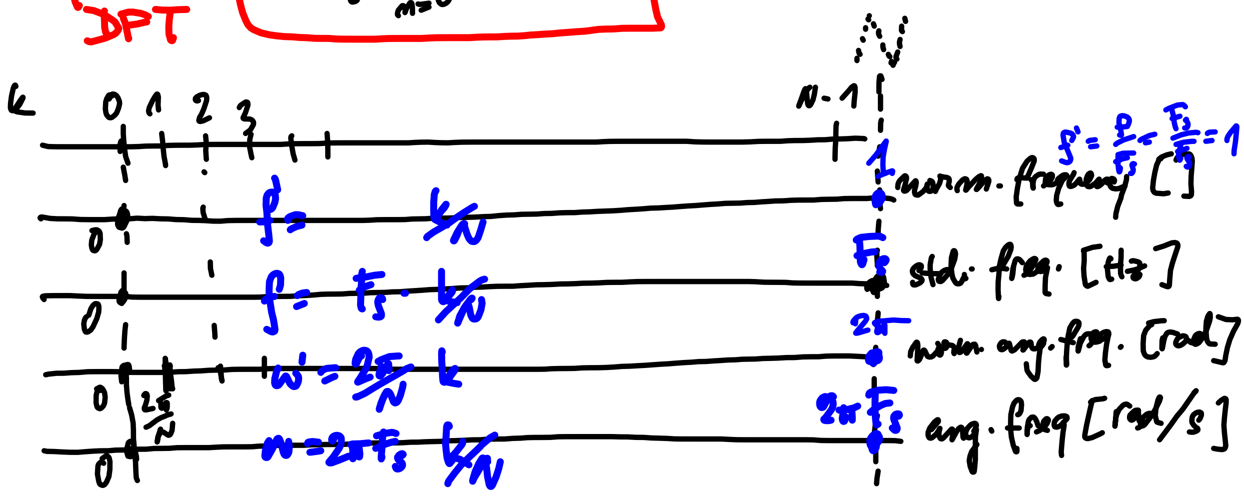
DFS:

$x[n] \Rightarrow \tilde{x}[n] = x[\mathrm{mod}_N(n)]$ $\quad \tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi}{N}kn} \longrightarrow \underline{X[k] = R_N[n]\tilde{X}[k]} \longrightarrow X[k]$

periodization! windowing

Discrete Fourier Transform DFT

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}} \qquad k \in 0 \dots N-1$$



DFT by hand

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi}{N}kn}=\sum x[n]e^{-j\frac{\pi}{2}kn}$$

	n	0	1	2	3	
	$x[n]$	2	2	0	0	
$k=0$	1	1	1	1	1	4
$k=1$ $e^{-j\frac{\pi}{2}1n}$		1	$-j$	-1	$+j$	$2-2j$
$k=2$ $e^{-j\frac{\pi}{2}2n}=e^{-j\pi n}$		1	-1	1	-1	0
$k=3$ $e^{-j\frac{\pi}{2}3n}$		1	j	-1	$-j$	$2+2j$

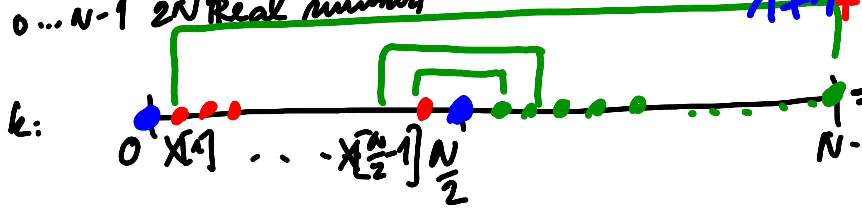
$k=1$, $k=3$

$$X[N-k]=X^*[k]$$

DFT - is it magically increasing information

$x[n]$ 0 … N-1 N Real numbers

$X[k]$ 0 … N-1 2N Real numbers



$$1+1+2\left(\frac{N}{2}-1\right)=2+N-2=\underline{N}$$

$X[0]=\sum x[n]e^{-j\frac{2\pi}{N}0n}=\sum x[n]$ 1 Real number!

$X[\frac{N}{2}]=X^*[N-\frac{N}{2}]=X^*[\frac{N}{2}]$ 1 Real number

$X[1]\dots X[\frac{N}{2}-1]$ Complex 2 Real numbers each $\frac{N}{2}-1$ numbers

$X[\frac{N}{2}+1]\dots X[N-1]$ complex conjugates of ↑

DFT of circular convolution

$x_1[n] \longrightarrow X_1[k]$

$x_2[n] \longrightarrow X_2[k]$

$x_1[n] \circledN x_2[n] \longrightarrow X_1[k] \cdot X_2[k]$

Remember

$x_1(t) \longrightarrow X_1(j\omega)$

$x_2(t) \longrightarrow X_2(j\omega)$

$x_1(t) * x_2(t) \longrightarrow X_1(j\omega) \cdot X_2(j\omega)$

$x_1[n] = [\,2 \quad 2 \quad 0 \quad 0\,]$ $\qquad x_2[n] = [\,1 \quad -1 \quad 0 \quad 0\,]$

	$X_1[k]$	$X_2[k]$	$X_1[k]X_2[k]$	DFT( )
k=0	4	0	0	0
1	2-2j	1+j	4	4
2	0	2	0	0
3	2+2j	1-j	4	4

$x_1[n] \circledN x_2[n] = [\,2 \quad 0 \quad -2 \quad 0\,]$

$\frac{fs}{2}$

$X[k] = \sum x[n] e^{-j\frac{2\pi}{N}kn}$ ops for one $X[k]$?

N mult + N additions complex!

FFT $N = 2^6$ 256, 512, 1024, 2048 all $X[k]$s? $O(N^2)$

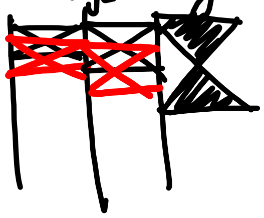
$O(N \log_2 N)$

same values

Cooley-Tukey

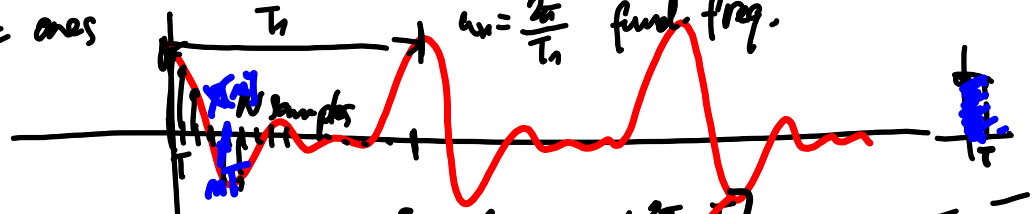
butterfly algorithm

$\log_2 N$ stages



~~Chap~~ Estimating spectra of continuous signals with DFT/FFT

periodic ones



$\omega_1 = \frac{2\pi}{T_1}$ fund. freq.

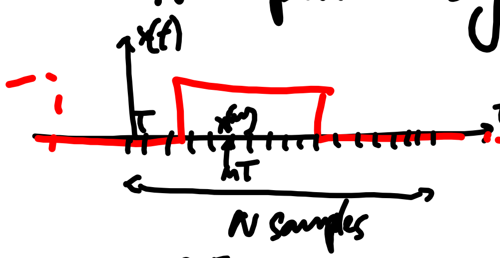
F.S.:

$$c_k = \frac{1}{T_1}\int_{T_1} x(t)\, e^{-jk\omega_1 t}\, dt = \frac{1}{N\cancel{T}}\left[\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N\cancel{T}} n\cancel{T}}\right]\cancel{T} = \frac{1}{N}\underbrace{\sum x[n]\, e^{-j\frac{2\pi}{N}kn}}_{X[k]} = \frac{1}{N}X[k]$$

BUT:

- sampling theo: $f_{max} < F_s/2$
- all $k = -\infty$ to ∞? NO, only from $-\frac{N}{2}$ to $\frac{N}{2}$
- $T_1 = N.T$ hard to fulfill! Don't know T_1! Use some usual N and usual F_s and see!

Non-periodic signal - approximating FT



$$\cancel{X(j\omega)} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \sum x[n]\, e^{-jk\frac{2\pi}{N\cancel{T}} n\cancel{T}} \cdot T =$$

$\frac{2\pi F_s}{N}$

$$X\left(jk\frac{2\pi F_s}{N}\right) = T \sum x[n]\, e^{-jk\frac{2\pi}{N}n} = T \cdot X[k]$$