ISS exercise #5 – Reference solution

1. $y[n] = [4 \quad 3 \quad 5 \quad 2 \quad -5]$
delay by 3 samples (linear one!)
$y_s[n] = [0 \quad 0 \quad 0 \quad 4 \quad 3 \quad 5 \quad 2 \quad -5]$

2. The modulo function produces
0 1 2 3 4 ~~5~~ 0 1 2 3 4 0 1 2 3 4 …
so that we have
$y_p[n] = [\underset{\dots}{4} \quad 3 \quad 5 \quad 2 \quad -5 \quad 4 \quad 3 \quad 5 \quad 2 \quad -5 \quad 4 \quad 3 \dots]$

3. Need to put this together with n's…

n	−5	−4	−3	−2	−1	0	1	2	3	4	5	6…
$y_{ps}[n]$	5	2	−5	4	3	5	2	−5	4	3	5	2

4. Window function cuts out samples
$n \in \langle 0 \dots N-1 \rangle$

n	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
$y_{psw}[n]$	0	0	0	0	0	5	2	−5	4	3	0	0

this is equal to a <u>circular shift</u>

5. Linear convolution works the same as we did it (use for example the strips of paper…). Result

n	0	1	2	3	4	5	6	7	8	9
$x_1[n] * x_2[n]$	4	7	12	10	2	−3	−5	0	0	0…

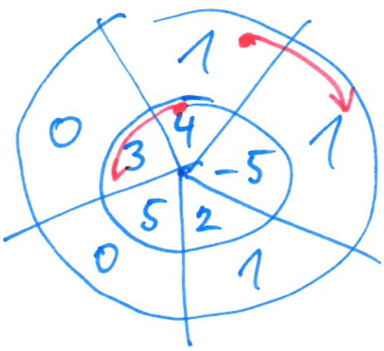
1 Compute circular convolution of two signals of length $N = 5$ samples:

n	0	1	2	3	4
$x[n]$	1	1	1	0	0
$y[n]$	4	3	5	2	-5

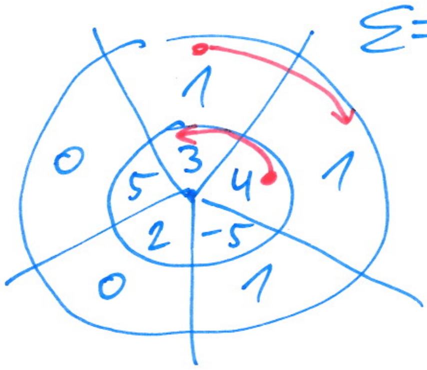
$$x[n] \circledast_N y[n] = \underbrace{R_N[n]}_{\text{windowing}} \sum_{k=0}^{N-1} x[k]\, y[\underbrace{\mathrm{mod}_N}_{\text{periodization}}\underbrace{(n-k)}_{\text{shift}}]$$

... Now, you can divide students into 5 groups and let each group compute one number!

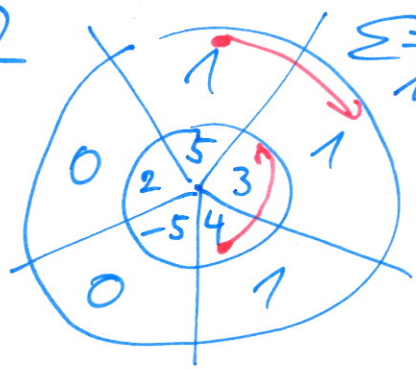
$n = 0$ $\Sigma = 1$



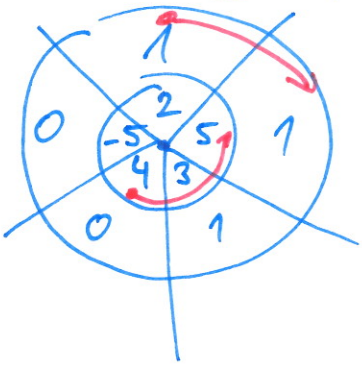
$n = 1$ $\Sigma = 2$



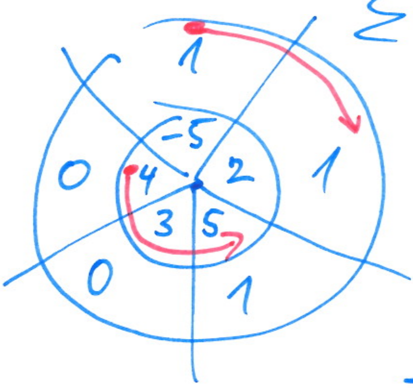
$n = 2$ $\Sigma = 12$



$n = 3$ $\Sigma = 10$



$n = 4$ $\Sigma = 2$



Result

1 2 12 10 2

7 The periodic convolution is the circular one

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7 ...
$x_1[n] \tilde{*} x_2[n]$	1	2	12	10	2	1	2	12	10	2	1	2	12 ...

(8) Discrete signal of length $N=4$ is given: (4)

n	0	1	2	3
$x[n]$	1	−1	0	0

Compute and <u>draw</u> its Discrete Time FT (DTFT). In Czech "Fourierova transformace s diskrétním časem" in interval $\omega \in \langle 0, 2\pi \rangle$

$$\tilde{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

DTFT is not limited to a given length of signal

<u>normalized</u> angular frequency

$$\tilde{X}(e^{j\omega}) = 1 \cdot \underbrace{e^{-j\omega 0}}_{1} + (-1)\, e^{-j\omega 1} = \underline{1 - e^{-j\omega}}$$

(9) We would end up here, but we want to draw it in magnitude and phase... Some work will be necessary to use

$$\sin\alpha = \frac{e^{j\alpha} - e^{-j\alpha}}{2j}$$

$$\tilde{X}(e^{j\omega}) = 1 - e^{-j\omega} = 2j\, e^{-j\frac{\omega}{2}} \frac{\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right)}{2j} =$$

$j = e^{j\frac{\pi}{2}}$

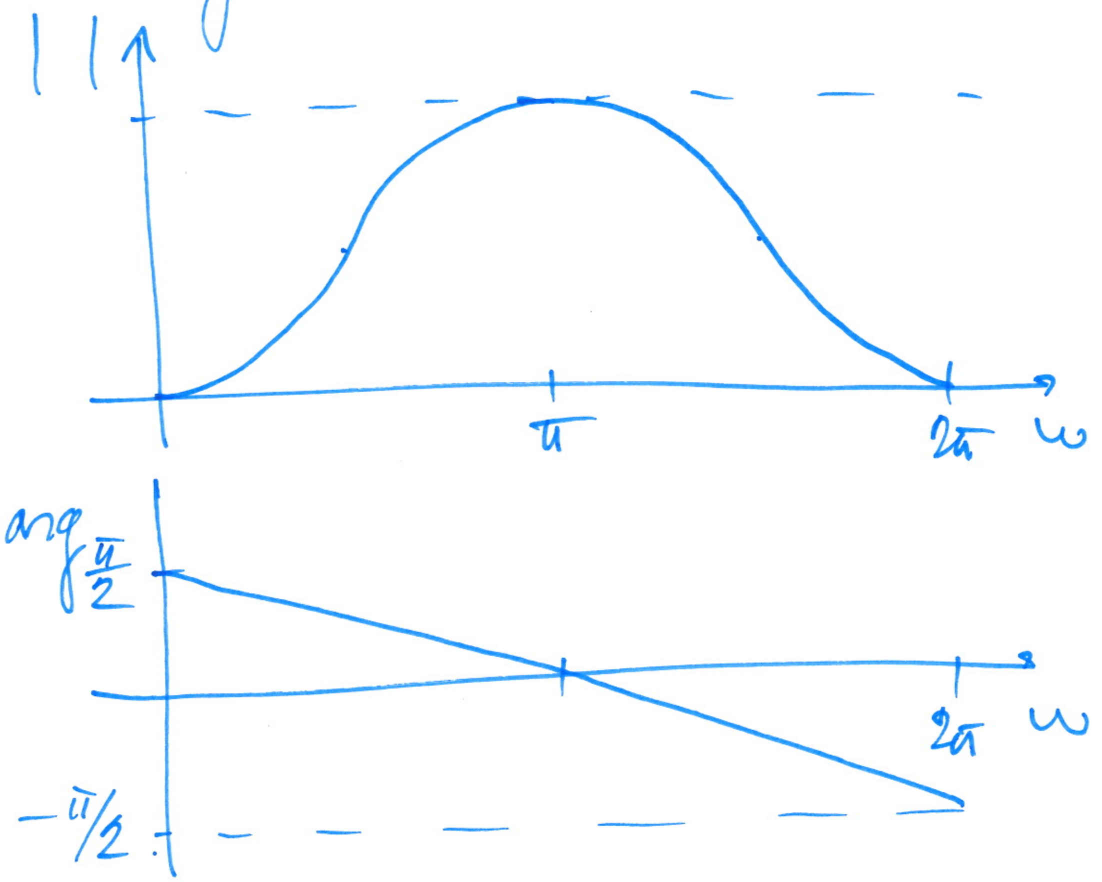
sin!

$$= 2e^{j\left(\frac{\pi}{2} - \frac{\omega}{2}\right)} \sin\frac{\omega}{2}$$

10 Magnitude will be $|\tilde{X}(e^{j\omega})| = 2\sin\frac{\omega}{2}$

11 Phase will be $\arg \tilde{X}(e^{j\omega}) = \underbrace{\frac{\pi}{2} - \frac{\omega}{2}}_{\text{simple linear function.}}$

Drawing them:



... look at Matlab plot :-)

12 Compute Discrete Fourier Transform (DFT) of the same signal.

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \quad \text{for } k \in \langle 0, N-1\rangle$$

For us: $\sum_{n=0}^{3} x[n]\, e^{-j\frac{2\pi}{4}kn} = \sum_{n=0}^{3} x[n]\, e^{-j\frac{\pi}{2}kn}$

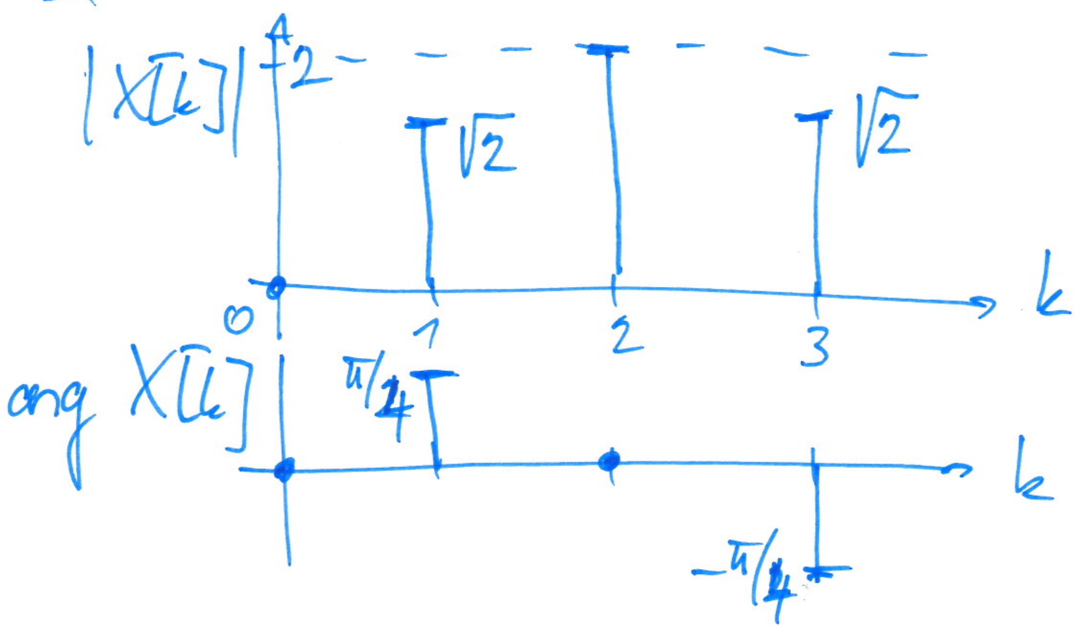
The rest is best shown on a table: (5)

(13)

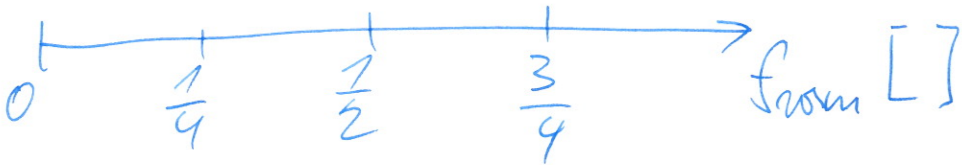
k	factors	m x[m]	0 1	1 −1	2 0	3 0	Σ
0	$e^{-j\frac{\pi}{2}\cdot 0\cdot m} = 1$		1	1	1	1	0 (14)
1	$e^{-j\frac{\pi}{2}1m}$		1	$-j$	-1	j	$1+j$ (15)
2	$e^{-j\frac{\pi}{2}2m} = e^{-j\pi m}$		1	-1	1	-1	2 (16)
3	$e^{-j\frac{3\pi}{2}m}$		1	j	-1	$-j$	$1-j$ (17)

(18)

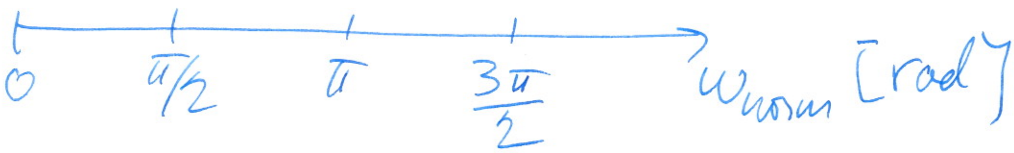
Draw it!



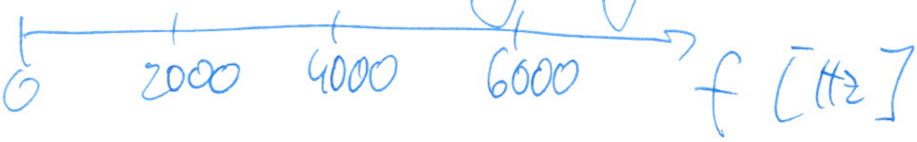
(19) Normalized frequencies are $\frac{k}{N}$



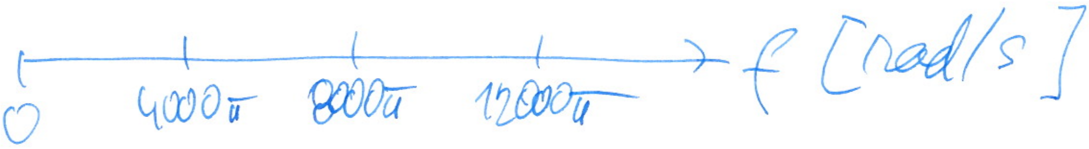
(20) Normalized angular frequencies are $2\pi\frac{k}{N}$



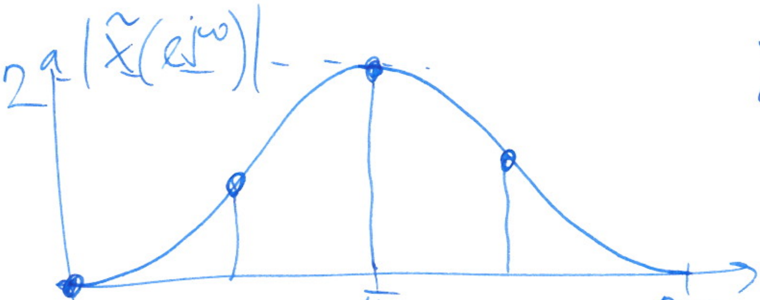
(21) De-normalizing by multiplying by F_s



(22) Multyplying by 2π or de-normalizing ω_{norm} ...



(23)

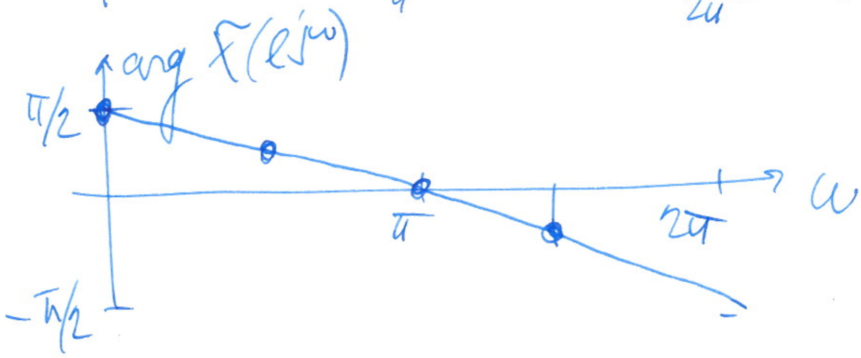


$X[0] = 0$

$X[1] = 1 + j$

$X[2] = 2$

$X[3] = 1 - j$



looks OK.

23 Check, that DFT actually <u>samples</u> DTFT at $\omega = k\frac{2\pi}{N}$

k	0	1	2	3
ω	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$

... looking at the figures: yes, Ok.

What about phase for $k=0$? ($\pi/2$ versus $\emptyset$?). We don't give a fuck about it, as magnitude is zero, so the phase can be anything.

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24 Compute DFT of a delayed signal:

$x'[n] = 0 \quad 1 \quad -1 \quad 0$. Yes it is a circular shift, $d=1$.

$X'[k] = X[k] \cdot e^{-j\frac{2\pi}{N}dk}$ ↑ delay. This holds only for circular delay, but we had zeros in our signal, so Ok.

$$X'[k] = X[k]e^{-j\frac{2\pi}{4}\cdot 1 \cdot k} =$$

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$$= X[k]e^{-j\frac{\pi}{2}k}$$

k	0	1	2	3
$e^{-j\frac{\pi}{2}k}$	1	$-j$	-1	j

(the same trick...)

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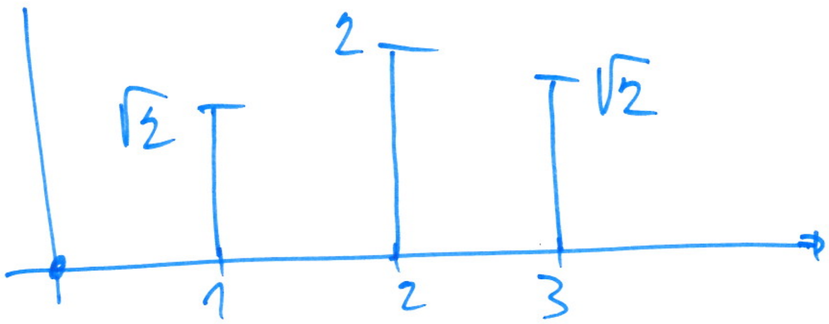
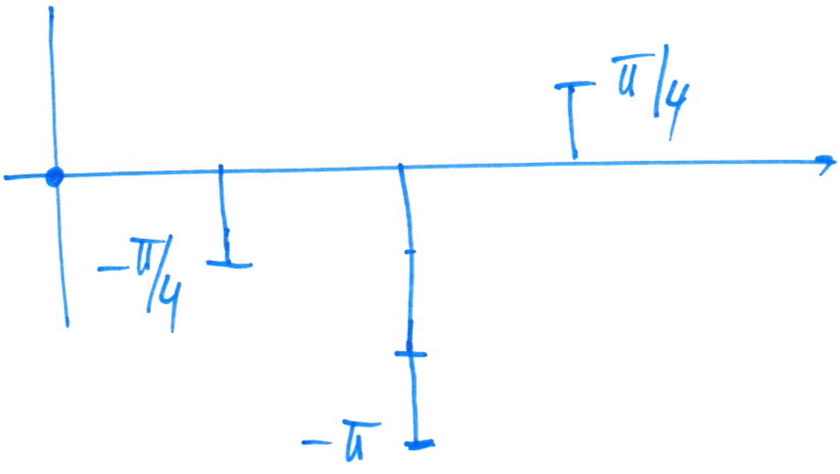
k	$e^{-j\frac{\pi}{2}k}$	$X[k]$	$X'[k]$
0	1	0	0
1	$-j$	$1+j$	$1-j$
2	-1	2	-2
3	j	$1-j$	$1+j$

Draw them:





... We could do the same by working directly on the argument — subtracting

$0 \qquad -\frac{\pi}{2} \qquad -\pi \qquad -\frac{3\pi}{2}$

here the result is $-\frac{7}{4}\pi$, we can flip it to $\frac{\pi}{4}$.

27 $X[17] = 2+j$

$X[256-17] = X^*[17]$

$X[239] = 2-j$

28 $X[k]$

0	1 - - - - - $N/2-1$	$N/2$	$N/2+1$	- - - -	$N-1$

real — complex — real — complex

conjugated (that means that $N/2+1 \ldots N+1$ are fully given by $1 \ldots N/2-1$.)

Amount of real numbers:

$1 + 2(N/2 - 1) + 1 = 1 + N - 2 + 1 = \underline{\underline{N}}$

(the same as amount of samples!)

29

0	1	$(N-1)/2$	$(N+1)/2$	$N-1$

real — complex — complex

conjugated

Amount of real numbers:

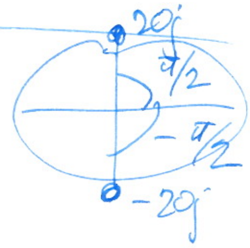
$1 + 2 \cdot \left(\frac{N-1}{2}\right) = 1 + N - 1 = N$ again!

n	0	1	2	3	4	5	6	7	$e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}kn}$
$5\cos\left(\frac{2\pi}{8}n\right)$	5	3,5	0	-3,5	-5	-3,5	0	3,5	
$5\cos\left(\frac{2\pi}{8}n + \frac{\pi}{2}\right)$	0	-3,5	-5	-3,5	0	3,5	5	3,5	$X[k]$
$k=0$ $e^{-j\frac{\pi}{4}0n}$	1	1	1	1	1	1	1	1	0
$k=1$ $e^{-j\frac{\pi}{4}1n}$	1	a-ja	-j	-a-ja	-1	-a+ja	+j	a+ja	$\cancel{5 + 3{,}5a - 3{,}5ja + 3{,}5a + 3{,}5ja + 5 + 3{,}5a - 3{,}5ja + 3{,}5a + 3{,}5ja}$ $0 \cancel{-3{,}5a} + 3{,}5ja + 5j \cancel{+3{,}5a} + 3{,}5ja + 0 -$ $\cancel{-3{,}5a} + 3{,}5ja + 5j \cancel{+3{,}5a} + 3{,}5ja =$ $= 10j + 14 \cdot \frac{1}{\sqrt{2}} j = 10j + 10j = 20j$
$k=2$ $e^{-j\frac{\pi}{4}2n}$	1	-j	-1	j	1	-j	-1	j	$0 \cancel{+3{,}5j} \cancel{+5} \cancel{-3{,}5j} + 0 \cancel{-3{,}5j} \cancel{-5} \cancel{+3{,}5j} =$ $= 0$
$k=3$ $e^{-j\frac{\pi}{4}3n}$	1	-a-ja	+j	a-ja	-1	a+ja	-j	-a+ja	$\cancel{3{,}5a} + 3{,}5ja - 5j \cancel{-3{,}5a} + 3{,}5ja + 0 \cancel{+3{,}5a} + 3{,}5ja$ $-5j \cancel{-3{,}5a} + 3{,}5ja = -10j + 14\frac{1}{\sqrt{2}}j =$ $= -10j + 10j = 0$
$k=4$ $e^{-j\frac{\pi}{4}4n} = e^{-j\pi n}$	1	-1	1	-1	1	-1	1	-1	$0 \cancel{+3{,}5} \cancel{-5} \cancel{+3{,}5} + 0 \cancel{-3{,}5} \cancel{+5} \cancel{-3{,}5} =$ $= 0$

3

31 cont the values $X[5, 6, 7]$ can be computed similarly or derived from the symmetry:

$$X[5] = X^*[3] = 0$$
$$X[6] = X^*[2] = 0$$
$$X[7] = X^*[1] = -20j$$



32 $|X[1]| = |X[7]| = 20$

this is $\frac{8 \cdot 5}{2} = \frac{40}{2} = 20 \Rightarrow$ Ok!

$\arg X[1] = \frac{\pi}{2}$ $\quad -\arg X[7] = \frac{\pi}{2}$

initial phase of the cosine was $\pi/2$

$\Rightarrow$ Ok!