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ELECTRONIC REPRESENTATION OF LINE REFLEX

ELEKTRONICKÁ REPRESENTACE ŘEŠENÍ ODRAZŮ NA VEDENÍ

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Abstract

It would be difficult to imagine a world without communication systems. In order to optimise guided communication systems, it is necessary to determine or project power and signal losses in the system, since all systems have such losses. To determine these losses and eventually ensure a maximum output, it is necessary to formulate some kind of equation with which to calculate these losses. A mathematical derivation for the telegraph equation in terms of voltage and current for a section of a transmission line will be investigated.

Abstrakt

Bylo by těžké si představit svět bez komunikačních systémů. Za účelem optimalizace řízených komunikačních systémů, je třeba určit signálové ztráty v systému, protože všechny systémy mají takové ztráty. Pro stanovení těchto ztrát a zajištění maximálního výkonu, je nutné formulovat určitý druh rovnice, kterým se tato ztráta vypočítá. Matematické odvození pro telegrafní rovnice, pokud jde o napětí a proud pro část přenosové linky bude předmětem zkoumání.

Keywords

Telegraph equations, transmission line, partial differential equations, Jacobian matrix, Taylor series, TKSL.

Klíčová slova

Telegrafní rovnice, přenosové vedení, parciální diferenciální rovnice, Jacobiho matice, Taylorovy řady, TKSL.

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Declaration

I declare that this bachelor thesis I developed separately under the leadership of *Mr. Jiří Kunovský*

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List of Abbreviations and Symbols

R	$[m/\Omega]$	Line resistance
L	$[H/m]$	Line inductance
C	$[F/m]$	Line capacitance
G	$[S/m]$	Line conductivity
Z_0	$[\Omega]$	Wave impedance
Z_p	$[\Omega]$	Output impedance
Z_k	$[\Omega]$	Input impedance
U_0	$[V]$	Voltage sources
U_p	$[V]$	Voltage at the beginning of the line
U_k	$[V]$	Voltage at the end of the line
I_p	$[A]$	Current at the beginning of the line
I_k	$[A]$	Current at the end of the line
f	$[Hz]$	Frequency
λ	$[m]$	Wavelength
l	$[m]$	Line length
u	$[V]$	Time-varying voltage
i	$[A]$	Time-varying Current
γ	$[S/m]$	conductivity
β	$[rad/m]$	Constant attenuation
v	$[m/s]$	Velocity of Propagation
ω	$[Hz]$	Angular frequency
N	–	Number of line elements
k	–	Order of the line elements
Δx	$[m]$	Spatial steps
Δt	$[s]$	Time steps
A	–	Jacobian matrix of constants
\vec{y}	–	vector of \vec{u}_c and \vec{i}_L
\vec{b}	–	vector of constants

Chapter 1

Introduction

A transmission line is a pair of electrical conductors carrying an electrical signal from one place to another.

Transmission lines are used for connecting radio transmitters and receivers with their antennas, distributing cable television signals, trunklines routing calls between telephone switching centres, computer network connections and high speed computer data buses and so on.

The first part of this thesis deals with homogeneous transmission lines parameters and telegraph equations, which can be described by a system of hyperbolic partial differential equations and describe current and voltage conditions on this line. Then it will present the conditions needed for the signal transfer without distortion (matched impedance) and perform numerical method on a specific examples.

In the second part the algorithm to solve and simulate the telegraph equation is investigated in MATLAB.

Third part is about presentation in MS PowerPoint which would describe the problematic of transmission lines. This presentation can be used for simply and quickly learning of the problem.

Chapter 2

Transmission line

This chapter is based on [14] [22] [13] [18] [9] [29] [12] [23].

A transmission line is a configuration of wires designed to guide electrical energy from one point to another. Transmission media can be categorised into two groups, namely *guided* and *unguided* transmission lines [6].

In a guided transmission media there will be some form of conductor that provides a conduit in which the signals are contained. Only devices that are physically connected to this conductor can receive the signals that propagate down the conductor. It is used, for example, to transfer the output rf energy of a transmitter to an antenna. This energy will not travel through normal electrical wire without great losses. The transmission line examples [22] is shown in the Figure 2.1

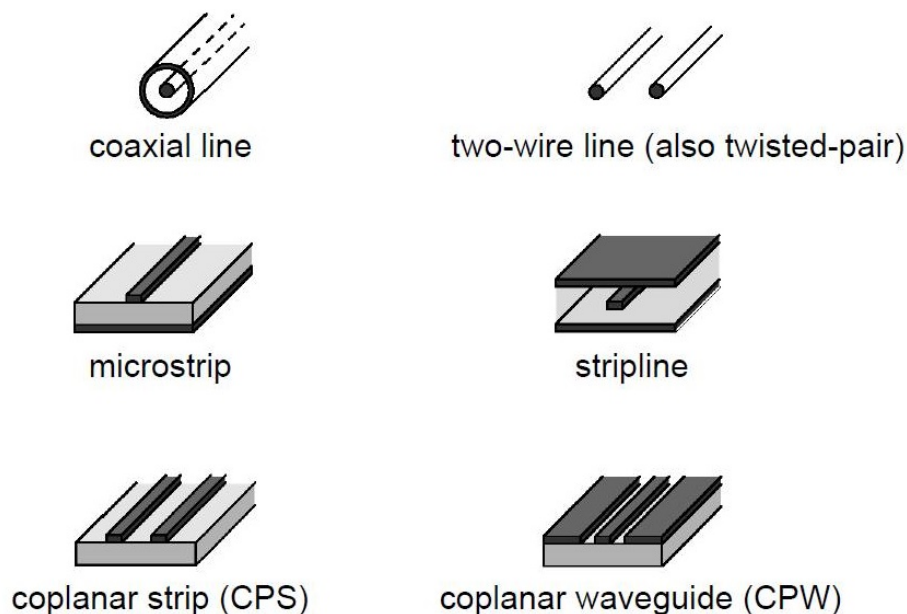


Figure 2.1: Transmission line examples.

Any two conductors can make up a transmission line. The signal which is transmitted from one end of the pair to the other end is the voltage between the conductors. Other

electrical elements which should be thought of as transmission lines include traces on printed circuit boards and multichip modules (MCMs) and within integrated circuits. With current technologies that use high-speed active devices on both ends of most circuit traces, all of the following transmission line effects must be considered during circuit analysis [24]

- Time delay
- Phase shift
- Power, voltage, and current loss
- Distortion
- Reduction of frequency bandwidth
- Coupled line crosstalk

2.1 Homogeneous Transmission Line

Homogeneous line is a transmission line where impedance is distributed uniformly on the whole length. A circuit containing homogenous transmission lines is often called “*distributed parameter circuit*”.

2.1.1 Lumped and distributed parameter circuits

In this subsection we address a subject in the field of distributed a lumped parameter circuits. The key difference between circuit theory and transmission line theory is line dimension vs wavelength.

There are two types of electrical circuits first one is distributed parameter circuit [22], in which

- Current varies along conductors and elements.
- Voltage across points along conductor or within element varies.
- Phase change or transit time cannot be neglected.
- The distributed model is used at high frequencies where the wavelength becomes comparable to the physical dimensions of the circuit, making the lumped model inaccurate.
- The distributed element model is more accurate but more complex than the lumped element model.

The second is the lumped parameter circuits, in which

- Physical dimensions of circuit are such that voltage across and current through conductors connecting elements does not vary.
- Current in two-terminal lumped circuit element does not vary (phase change or transit time are neglected).

For accurate modeling of the transmission line we must not assume that the parameters are lumped but are distributed throughout line, where voltages and currents can vary in magnitude and phase over its length.

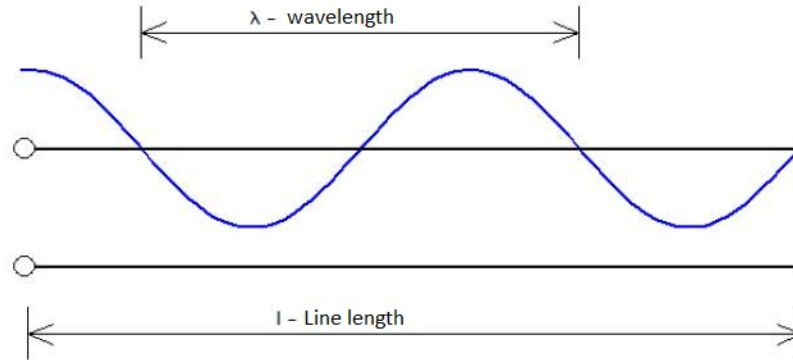


Figure 2.2: line length and wave length.

Example 1 *distributed or lumped circuit*

Coaxial cable with length 2 m is connected to harmonic sources with a frequency of 1 kHz

Frequency 1 kHz corresponds to a wavelength

$$\lambda = \frac{v}{f} = \frac{3 \cdot 10^8}{10^3} = 3 \cdot 10^5 m$$

in this case we can solve it as a **lumped circuit**.

Coaxial cable with length 2 m is connected to harmonic sources with a frequency of 100 MHz

Frequency 100 MHz corresponds to a wavelength

$$\lambda = \frac{v}{f} = \frac{3 \cdot 10^8}{10^6} = 3 m$$

in this case we can solve it as a **distributed circuit**.

2.2 Transmission line theory

The theory of the transmission lines is briefly discussed in this section. A transmission line is a two-port network connecting a generator circuit at the sending end to a load at the receiving end. Proper understanding and interpretation of transmission line behavior is essential to the design and analysis of interconnectivity in high speed integrated circuits.

As shown in the Figure 2.2, a transmission line is often schematically represented as a two-wire line. The piece of the line of infinitesimal length can be modeled as an element circuit, as shown in the Figure 2.3, where R, L, G, C are per unit length quantities defined as follows:

- R : series resistance per unit length, for both conductors, in Ω/m .
- L : series inductance per unit length, for both conductors, in H/m .
- G : shunt conductance per unit length, in S/m .
- C : shunt capacitance per unit length, in F/m .

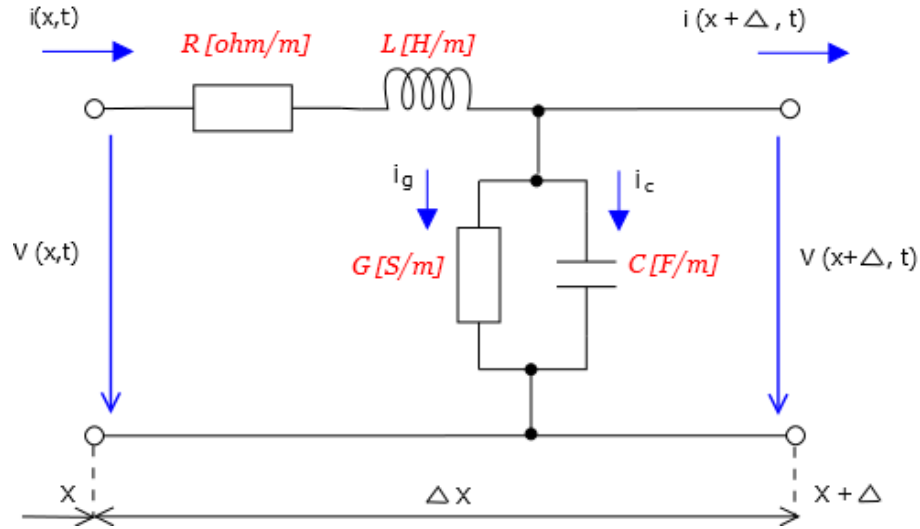


Figure 2.3: Element of the transmission line.

The parameters need to be calculated or estimated before any line model can be built. In particular, inductance and capacitance depend on the line and tower geometry. The series inductance L represents the total self-inductance of the two conductors, and the shunt capacitance C is due to the close proximity of the two conductors. The series resistance R represents the resistance due to the finite conductivity of the conductors, and the shunt conductance G is due to dielectric loss in the material between the conductors. R and G , therefore, represent losses. Conductance G is ignored in short circuit studies because the inductance of the line is the dominant value. Conductance may not be ignored in stability studies. A finite length of transmission line can be viewed as a cascade of sections (Figure 2.4)

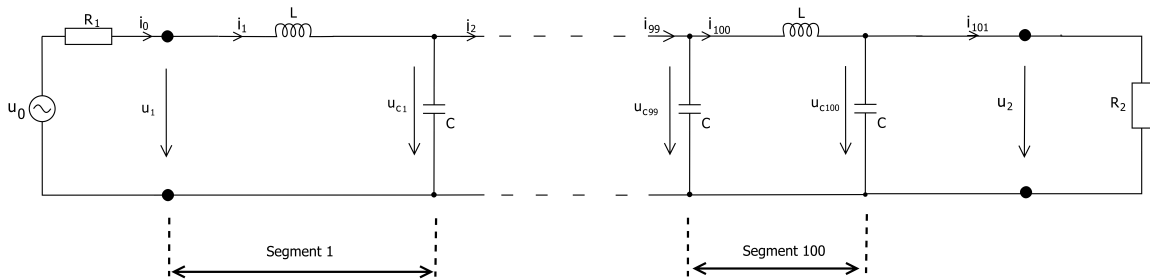


Figure 2.4: Equivalent circuit of transmission line.

In this section I presented the basic information for transmission line to understand the **Telegrapher's equation** below:

$$-\frac{\partial^2 v}{\partial x^2} = R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t}$$

$$-\frac{\partial^2 i}{\partial x \partial t} = G \frac{\partial v}{\partial t} + C \frac{\partial^2 v}{\partial t^2}$$

Telegrapher's equation will be discussed in subsection **2.3.2**

2.3 Transmission Line Equations

Consider a piece of wire being modeled as an electrical circuit element (see Figure **2.3**) consisting of the infinitesimal piece of wire with resistance $R\Delta x$ and inductance $L\Delta x$, while it is connected to a ground with conductance $(G\Delta x)^{-1}$ and capacitance $C\Delta x$ (where x is finite length of the transmission line). Let $i(x; t)$ and $v(x; t)$ denote the current and voltage across the piece of wire at position x at time t [29]. From the circuit in the Figure **2.3**, Kirchhoff's law can be applied as will be shown in the next section.

2.3.1 Kirchhoff's circuit laws

In complex circuits, we can not simply use Ohm's Law alone to find the voltages or currents circulating within the circuit. For these types of calculations we need certain rules which allow us to obtain the circuit equations and for this we can use Kirchhoff's Circuit Law.

In 1845, a German physicist, Gustav Kirchhoff developed a pair or set of rules or laws which deal with the conservation of current and voltage within Electrical Circuits. These two rules are commonly known as: Kirchhoff's Circuit Laws with one of Kirchhoff's laws dealing with the current flowing around a closed circuit, *Kirchhoff's Current Law (KCL)* while the other law deals with the voltage sources present in a closed circuit, *Kirchhoff's Voltage Law (KVL)*.

a) Kirchhoff's First Law (KCL) – Kirchhoff's Current Law or KCL, states that the „total current or charge entering a junction or node is exactly equal to the charge leaving the node as it has no other place to go except to leave, as no charge is lost within the node“. In other words the algebraic sum of all the currents entering and leaving a node must be equal to zero, $I(\text{exiting}) + I(\text{entering}) = 0$. This idea by Kirchhoff is commonly known as the *Conservation of Charge*.

Applying Kirchhoff's current law to Figure **2.5** circuit gives the following equations:

$$i(x, t) = v(x, t)G\Delta x + \frac{\partial v}{\partial t}(x, t)C\Delta x + i(x + \Delta, t)$$

$$i(x + \Delta, t) - i(x, t) = -v(x, t)G\Delta x - \frac{\partial v}{\partial t}(x, t)C\Delta x \quad (2.3.1)$$

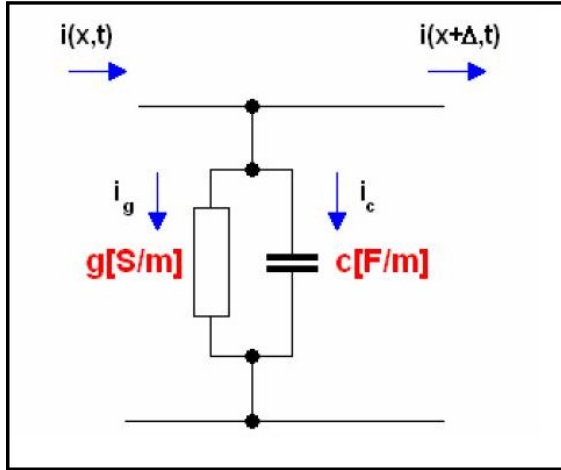


Figure 2.5: currents flowing through an element.

a) Kirchoffs Second Law(KVL) – Kirchoffs Voltage Law or KVL, states that „in any closed loop network, the total voltage around the loop is equal to the sum of all the voltage drops within the same loop“ which is also equal to zero. In other words the algebraic sum of all voltages within the loop must be equal to zero. This idea by Kirchoff is known as the *Conservation of Energy*.

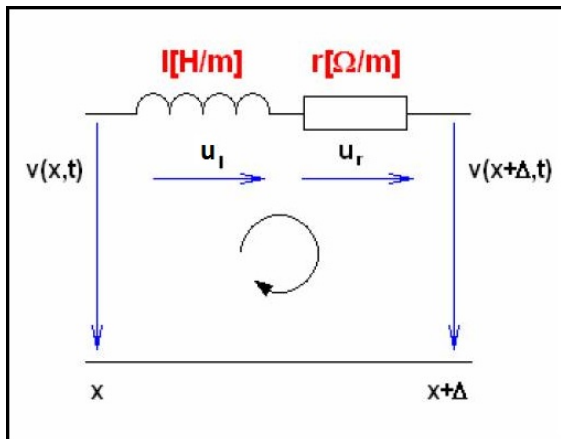


Figure 2.6: Voltage flowing through an element.

Applying Kirchoff's current law to Figure 2.6 circuit gives the following equations:

$$v(x, t) = i(x, t)R\Delta x + \frac{\partial i}{\partial t}(x, t)L\Delta x + v(x + \Delta, t)$$

$$v(x + \Delta, t) - v(x, t) = -i(x, t)R\Delta x - \frac{\partial i}{\partial t}(x, t)L\Delta x \quad (2.3.2)$$

2.3.2 Telegrapher's equation

This section provides slightly more detail about how we derive the transmission line equations. It's well known that voltage and current changes on the line due to the change in

distance and time from the source x , so at a distance x from the source u and i have value $u = u(x, t)$ and $i = i(x, t)$.

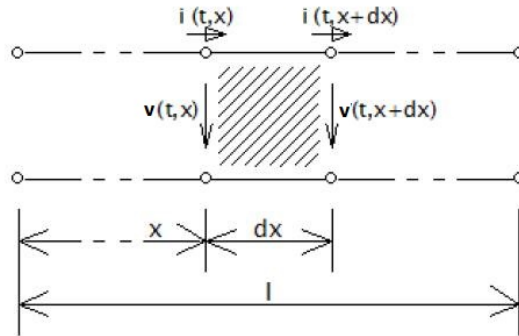


Figure 2.7: demarcation elements.

Therefore the voltage and current at a distance $v(x + dx, t)$ and $i(x + dx, t)$ is solved by *Taylor Series*, in which the second and higher derivation is neglected:

$$v(x + dx, t) = v(x, t) + \frac{\partial v}{\partial x} dx \quad (2.3.3)$$

$$i(x + dx, t) = i(x, t) + \frac{\partial i}{\partial x} dx \quad (2.3.4)$$

The telegrapher's equations is the pair of coupled, linear differential equations that describe the voltage and current on an electrical transmission line with distance and time.

When the elements R and G are very small, their effects can be neglected, and the transmission line is considered as an ideal lossless structure. In this case, the model depends only on the L and C elements. The equations themselves consist of a pair of coupled, first-order, partial differential equations. The first equation (2.3.5) shows that the induced voltage is related to the time rate-of-change of the current through the cable inductance, while the second equation (2.3.6) shows, similarly, that the current drawn by the cable capacitance is related to the time rate-of-change of the voltage.

$$\frac{\partial v}{\partial x} = -L \frac{\partial i}{\partial t} \quad (2.3.5)$$

$$\frac{\partial i}{\partial x} = -C \frac{\partial v}{\partial t} \quad (2.3.6)$$

When the loss elements R and G are not negligible, the original differential equations describing the elementary segment of line become after modifying equations (2.3.1) and (2.3.2) and dividing by Δx and letting $\Delta x \rightarrow 0$

$$-\frac{\partial v}{\partial x} = Ri + L \frac{\partial i}{\partial t} \quad (2.3.7)$$

$$-\frac{\partial i}{\partial x} = Gv + C \frac{\partial v}{\partial t} \quad (2.3.8)$$

To get the voltage equation we have to derive the equation (2.3.7) by x and to get the current equation we have to derive the equation (2.3.8) by t

$$-\frac{\partial^2 v}{\partial x^2} = R\frac{\partial i}{\partial x} + L\frac{\partial^2 i}{\partial x \partial t} \quad (2.3.9)$$

$$-\frac{\partial^2 i}{\partial x \partial t} = G\frac{\partial v}{\partial t} + C\frac{\partial^2 v}{\partial t^2} \quad (2.3.10)$$

After differentiating both equations with respect to x , and some algebraic manipulation, we obtain a pair of hyperbolic partial differential equations each involving only one unknown (by substitution of equations (2.3.8) and (2.3.10) in equation (2.3.9)). These equations are referred to as the **telegrapher's equations**.

$$-\frac{\partial^2 v}{\partial x^2} = GRv + (RC + GL)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \quad (2.3.11)$$

The same way for current:

$$-\frac{\partial^2 i}{\partial x^2} = GRi + (RC + GL)\frac{\partial i}{\partial t} + LC\frac{\partial^2 i}{\partial t^2} \quad (2.3.12)$$

Note that these equations resemble the homogeneous wave equation with extra terms for v and i and their first derivatives. These extra terms cause the signal to decay and spread out with time and distance.

2.4 Harmonic steady state on the transmission line

If the transmission line is in the harmonic steady state, the solution of the differential equations is simpler [23]. Time variable harmonic voltage and current can be expressed as

$$u(x, t) = U_m(x)\sin[\omega t + \psi_u(x)] \quad (2.4.1)$$

$$i(x, t) = I_m(x)\sin[\omega t + \psi_i(x)] \quad (2.4.2)$$

can be replaced by phasors

$$\hat{U}(x) = U(x)e^{j\psi_u(x)} \quad (2.4.3)$$

$$\hat{I}(x) = I(x)e^{j\psi_i(x)} \quad (2.4.4)$$

and the basic equations take form of the ordinary differential equations

$$-\frac{d\hat{U}(x)}{dx} = \hat{Z}_0\hat{I}(x) \quad (2.4.5)$$

$$-\frac{d\hat{I}(x)}{dx} = \hat{Y}_0 \hat{U}(x) \quad (2.4.6)$$

Derivatives with respect to time are replaced by multiplying $j\omega$, which are added to *longitudinal specific impedance*

$$\hat{Z}_0 = R_0 + j\omega L_0 \quad (2.4.7)$$

with unit (Ω/m) and *specific transverse admittance*

$$\hat{Y}_0 = G_0 + j\omega C_0 \quad (2.4.8)$$

with unit S/m

Telegrapher equation for voltage is obtained by derivation equation (2.4.5) with respect to x and substitution in second equation to exclude the current. Similar procedure the telegraph equation for current is derived.

$$\frac{d^2 \hat{U}}{dx^2} = \hat{Z}_0 \hat{Y}_0 \hat{U} = \hat{\gamma}^2 \hat{U} \quad (2.4.9)$$

$$\frac{d^2 \hat{I}}{dx^2} = \hat{Z}_0 \hat{Y}_0 \hat{I} = \hat{\gamma}^2 \hat{I} \quad (2.4.10)$$

specific propagation factor

$$\hat{\gamma} = \beta + j\alpha = \sqrt{\hat{Z}_0 \hat{Y}_0} = \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)} \quad (2.4.11)$$

Its real part β - *specific attenuation* - shows a change in signal voltage (current) amplitude and imaginary part shows signal phase shift along transmission line.

The characteristic equation of the transmission line can be written

$$\lambda^2 - \hat{\gamma}^2 = 0 \quad (2.4.12)$$

To find its roots

$$\lambda_{1,2} = \pm \hat{\gamma} \quad (2.4.13)$$

Solving for voltage

$$\hat{U}(x) = \hat{A}e^{-\hat{\gamma}x} + \hat{B}e^{\hat{\gamma}x} \quad (2.4.14)$$

Integration constants \hat{A}, \hat{B} , which depend on the initial conditions. From (2.4.5) the current can be calculated as

$$\hat{I}(x) = -\frac{1}{\hat{Z}_0} \frac{d\hat{U}(x)}{dx} \quad (2.4.15)$$

Substitute voltage by x can be obtain

$$\hat{I}(x) = -\frac{1}{\hat{Z}_v} (\hat{A}e^{-\hat{\gamma}x} - \hat{B}e^{\hat{\gamma}x}) \quad (2.4.16)$$

Wave impedance is described by

$$\hat{Z}_v = \sqrt{\frac{\hat{Z}_0}{\hat{Y}_0}} = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} \quad (2.4.17)$$

Wave impedance and propagation factor are called *secondary parameters of transmission line*. Integration constants can be calculated in case that the voltage and current values are known values at the beginning of transmission line. From conditions

$$\hat{U}(0) = \hat{U}_1, \hat{I}(0) = \hat{I}_1 \quad (2.4.18)$$

The integral constants can be calculated

$$\hat{A} = \frac{1}{2}(\hat{U}_1 + \hat{Z}_v \hat{I}_1), \hat{B} = \frac{1}{2}(\hat{U}_1 - \hat{Z}_v \hat{I}_1) \quad (2.4.19)$$

After substituting and some treatment we obtain

$$\hat{U}(x) = \frac{\hat{U}_1 + \hat{Z}_v \hat{I}_1}{2} e^{-\hat{\gamma}x} + \frac{\hat{U}_1 - \hat{Z}_v \hat{I}_1}{2} e^{\hat{\gamma}x} \quad (2.4.20)$$

$$\hat{I}(x) = \frac{\hat{U}_1 + \hat{Z}_v \hat{I}_1}{2\hat{Z}_v} e^{-\hat{\gamma}x} + \frac{\hat{U}_1 - \hat{Z}_v \hat{I}_1}{2\hat{Z}_v} e^{\hat{\gamma}x} \quad (2.4.21)$$

By hyperbolic function we obtain the equations

$$\hat{U}(x) = \hat{U}_1 \cosh \hat{\gamma} x - \hat{Z}_v \hat{I}_1 \sinh \hat{\gamma} x \quad (2.4.22)$$

$$\hat{I}(x) = -\frac{\hat{U}_1}{\hat{Z}_v} \sinh \hat{\gamma} x + \hat{I}_1 \cosh \hat{\gamma} x \quad (2.4.23)$$

Voltage at distance x is given by sum of onward and reflected harmonic voltage waves

$$\hat{U}(x) = \hat{U}_{p1} e^{-\hat{\gamma}x} + \hat{U}_{z1} e^{\hat{\gamma}x} = \hat{U}_p(x) + \hat{U}_z(x) \quad (2.4.24)$$

When onward and reflected harmonic voltage waves at the beginning of transmission line are given

$$\hat{U}_{p1} = \hat{U}_{p1} e^{-\hat{\gamma} \psi_{up1}} = \frac{\hat{U}_1 + \hat{Z}_v \hat{I}_1}{2} \quad (2.4.25)$$

$$\hat{U}_{z1} = \hat{U}_{z1} e^{-\hat{\gamma} \psi_{uz1}} = \frac{\hat{U}_1 - \hat{Z}_v \hat{I}_1}{2} \quad (2.4.26)$$

Also current anywhere at transmission line is given by sum of the onward

$$\hat{I}_p(x) = \frac{\hat{U}_p(x)}{\hat{Z}_v} = \frac{\hat{U}_{p1}}{\hat{Z}_v} e^{-\hat{\gamma}x} \quad (2.4.27)$$

and the reflected harmonic current wave

$$\hat{I}_z(x) = -\frac{\hat{U}_z(x)}{\hat{Z}_v} = -\frac{\hat{U}_{z1}}{\hat{Z}_v} e^{\hat{\gamma} x} \quad (2.4.28)$$

Telegraph equation can be solved by similar procedure for known values voltage and current at the end of transmission line. Output is equation in which the voltage and current can be determined at the distance x from transmission lines end.

$$\hat{U}(y) = \frac{\hat{U}_2 + \hat{Z}_v \hat{I}_2}{2} e^{\hat{\gamma} y} + \frac{\hat{U}_2 - \hat{Z}_v \hat{I}_2}{2} e^{-\hat{\gamma} y} = \hat{U}_{p2} e^{\hat{\gamma} y} + \hat{U}_{z2} e^{-\hat{\gamma} y} \quad (2.4.29)$$

$$\hat{I}(y) = \frac{\hat{U}_2 + \hat{Z}_v \hat{I}_2}{2 \hat{Z}_v} e^{\hat{\gamma} y} - \frac{\hat{U}_2 - \hat{Z}_v \hat{I}_2}{2 \hat{Z}_v} e^{-\hat{\gamma} y} = \frac{\hat{U}_{p2}}{\hat{Z}_v} e^{\hat{\gamma} y} - \frac{\hat{U}_{z2}}{\hat{Z}_v} e^{-\hat{\gamma} y} \quad (2.4.30)$$

First part of telegraph equation (2.4.29) and (2.4.30) presents the onward wave, the second part presents the reflected wave. Using the relationship between exponential and hyperbolic functions, the equations can be written as

$$\hat{U}(y) = \hat{U}_2 \cosh \hat{\gamma} y + \hat{Z}_v \hat{I}_2 \sinh \hat{\gamma} y \quad (2.4.31)$$

$$\hat{I}(y) = \frac{\hat{U}_2}{\hat{Z}_v} \sinh \hat{\gamma} y + \hat{I}_2 \cosh \hat{\gamma} y \quad (2.4.32)$$

Decomposition to onward and reflected components facilitates analysis voltage and current rates at transmission line.

Example 2 *Homogenic transmission line with primary parameters $R_0 = 0.1\Omega/m$, $G_0 = 5 \times 10^{-8} S/m$, $C_0 = 50 pF/m$, $L_0 = 0.25 \mu H/m$ working at frequency $f = 15.915 MHz$. transmission line is loaded with wave impedance. Determine secondary parameters of transmission line and input voltage, if the output voltage is $U_2 = 10V$*

The angular frequency can be calculated as

$$\omega = 2\pi f = 2\pi \times 15.915 \times 10^6 = 1 \times 10^8$$

Specific impedance and admittance can be calculated

$$\hat{Z}_0 = R_0 + j\omega L_0 = 0.1 + j1 \times 10^8 \times 0.25 \times 10^{-6} = 0.1 + j25 [\Omega/m]$$

$$\hat{Y}_0 = G_0 + j\omega C_0 = 5 \times 10^{-8} + j50 \times 10^{-12} = 5 \times 10^{-8} + j5 \times 10^{-3} [S/m]$$

Next calculation the G_0 is neglected then

$$\hat{Z}_v = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} = \sqrt{\frac{0.1 + j25}{j5 \times 10^{-3}}} = 70.711 \angle -0.1146^\circ$$

Propagation factor is

$$\hat{\gamma} = \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)} = \sqrt{(0.1 + j 25)(j 5 \times 10^{-3})} = 0.707 \times 10^{-3} + j0.3535$$

In matched transmission line

$$\hat{U}_1 = \hat{U}_2 e^{\hat{\gamma}l} = \hat{U}_2 e^{\beta l} e^{j\alpha}$$

Then from this relationship the voltage on input and output is

$$U_1 = U_2 e^{\beta l} = 10 e^{0.707 \times 10^{-3} \times 100} = 10.732V$$

2.5 Wave propagation on the transmission line

In the previous chapter the equations were derived, in which by phasors (complex effective value) can capture the voltage and current rates on transmission line at time $t = 0$. From these equations the changes in effective values and the initial phase voltages and currents can be traced. For exploring propagation of waves on the transmission line it is necessary to take into account the instant voltage and current values [23].

We will investigate voltage wave, for which the following equation holds

$$\hat{U}(x) = \hat{U}_{p1} e^{-\gamma x} + \hat{U}_{z1} e^{\gamma x} \quad (2.5.1)$$

voltage in equation (2.5.1) can be expressed in complex immediate values and by substituting $\hat{\gamma} = \beta + j\alpha$ to exponential functions we obtain

$$\hat{u}(x, t) = U_{mp1} e^{-\beta x} e^{j(\omega t + \psi_{up1} - \alpha x)} + U_{mz1} e^{\beta x} e^{j(\omega t + \psi_{uz1} + \alpha x)} \quad (2.5.2)$$

The voltage immediate value is an imaginary part of complex immediate values,

$$u(x, t) = U_{mp1} e^{-\beta x} \sin(\omega t + \psi_{up1} - \alpha x) + U_{mz1} e^{\beta x} \sin(\omega t + \psi_{uz1} + \alpha x) \quad (2.5.3)$$

First part of the equation (2.5.3) presents the successive wave with an amplitude decreasing exponentially with distance x and phase dependent on the time t and the distance x . The presentation for two of consecutive successive waves is shown in the Figure 2.8

we see that it is a damped harmonic wave progressing in the direction of the x-axis. progression characterizes the pace of the place with a constant phase called *phase velocity*.

$$v_f = \frac{dx}{dt} = \frac{\omega}{\alpha} \quad (2.5.4)$$

The distance traveled by the wave during a single cycle is called wavelength,

$$\lambda = v_f T = \frac{v_f}{f} = \frac{2\pi}{\alpha} \quad (2.5.5)$$

Second part of the equation (2.5.3) presents reflected damped wave moving with phase velocity

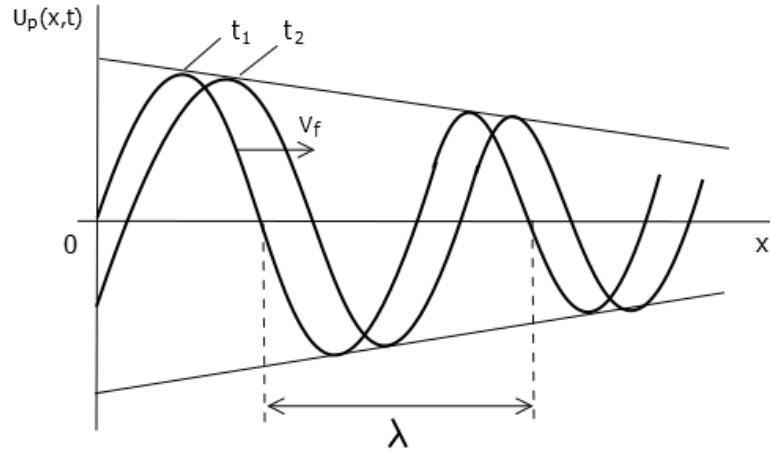


Figure 2.8: Successive wave with an amplitude decreasing exponentially.

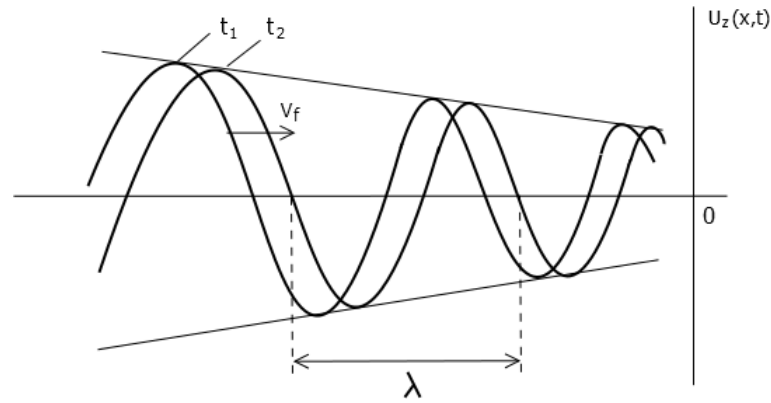


Figure 2.9: Successive wave with an amplitude increasing exponentially.

$$v_f = -\frac{\omega}{\alpha} \quad (2.5.6)$$

The presentation for two in consecutive successive waves is shown in the Figure 2.9.

From the equation (2.5.3) it is obvious that the voltage on transmission line is given by sum of the successive wave and the reflected wave

Similarly such as voltage, current instantaneous value can be derived.

$$u(x, t) = \frac{U_{mp1}}{Z_v} e^{-\beta x} \sin(\omega t + \psi_{up1} - \varphi_v - \alpha x) + \frac{U_{mz1}}{Z_v} e^{\beta x} \sin(\omega t + \psi_{uz1} - \varphi_v + \alpha x) \quad (2.5.7)$$

From this equation it is clear, that the currents wave propagates in transmission line similarly as the voltages wave. Currents wave is shifted to the argument wave impedance φ_v , reflected wave is in antiphase.

2.6 Reflection coefficient

Reflection coefficient is the physical ratio, which describes a ratio between a voltage of the successive wave travelling from the beginning to the end of transmission line and the reflected voltage from the end of the transmission line [23]. The reflected wave propagates to the beginning of transmission line. Reflection at the end affected by loading impedance.

$$\hat{Z}_2 = \frac{\hat{U}_2}{\hat{I}_2} \quad (2.6.1)$$

Reflection coefficient is defined by ratio reflected to successive voltage wave components at the end of transmission line and given

$$\hat{\rho}_{u2} = \frac{\hat{U}_{z2}}{\hat{U}_{p2}} \quad (2.6.2)$$

reflected and successive voltage wave components at the end of transmission line are obtained from the equation (2.4.29) when $y = 0$

$$\hat{\rho}_{u2} = \frac{\frac{\hat{U}_2 - \hat{Z}_v \hat{I}_v}{2}}{\frac{\hat{U}_2 + \hat{Z}_v \hat{I}_v}{2}} = \frac{\frac{\hat{U}_2}{\hat{I}_2} - \hat{Z}_v}{\frac{\hat{U}_2}{\hat{I}_2} + \hat{Z}_v} = \frac{\hat{Z}_2 - \hat{Z}_v}{\hat{Z}_2 + \hat{Z}_v} \quad (2.6.3)$$

and for reflection coefficient current wave we obtain

$$\hat{\rho}_{i2} = -\frac{\hat{Z}_2 - \hat{Z}_v}{\hat{Z}_2 + \hat{Z}_v} \quad (2.6.4)$$

from the last relations follows

$$\hat{\rho}_{u2} = -\hat{\rho}_{i2} \quad (2.6.5)$$

Reflection coefficient is a complex number, therefore the reflected voltage and current changes its value and phase. In a many applications, the transmission line without reflection is always required. As shown by the relations (2.6.3) it can be obtained by adding impedance to transmission line

$$\hat{Z}_2 = -\hat{Z}_v \quad (2.6.6)$$

Transmission line loaded with wave impedance is called matched transmission line where are no reflection $\hat{\rho}_{u2} = 0$. maximum value of reflection coefficient is derived by extreme value loaded impedance:

For short Transmission line ($\hat{Z}_2 = 0$) :

- reflected voltage has a same amplitude, then opozit phase than successive voltage, then the total voltage $U_2 = 0$
- reflected current is in phase, then total current is doubly than successive current.

For no-load transmission line ($\hat{Z}_2 \rightarrow \infty$) :

- reflected voltage is in phase, then total voltage is doubly than successive voltage
- reflected current is not in phase, then total current is zero ($\hat{I}_2 = 0$).

Reflection coefficient at distance y is defined as the ratio of reflected and successive wave at distance y from the end of transmission line

$$\hat{\rho}(y) = \frac{\hat{U}_z(y)}{\hat{U}_p(y)} = \hat{\rho}_2 e^{-2\hat{\gamma} y} \quad (2.6.7)$$

2.7 Transmission properties of transmission line

Before using transmission line to transfer energy or signals it is necessary to know the relations between input and output quantities [23]. This ratio affects impedance ratios at input, output of transmission line and applied frequency.

2.7.1 Transmission line as transmission cell

Typical transmission line is represented in the Figure 2.10. Transmission line presents transmission cell that has a power source with voltage \hat{U}_i , input impedance \hat{Z}_i and output impedance \hat{Z}_2 . As it is shown from input-output voltages and currents, the transmission line can be presented as two-port (Figure 2.11).

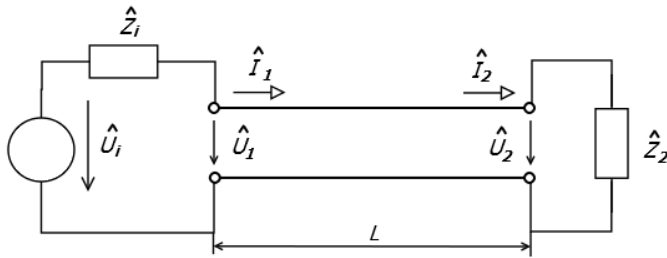


Figure 2.10

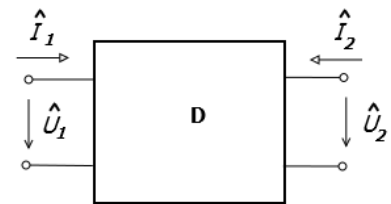


Figure 2.11

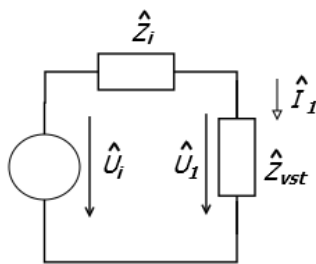


Figure 2.12

Investigated ratios the input of transmission line can perform using schema in the Figure 2.12 in which the transmission line is loaded by impedance \hat{Z}_2 , then input impedance is:

$$\hat{Z}_{inp} = \frac{\hat{U}_1}{\hat{I}_1} \quad (2.7.1)$$

Input voltage and current can be determined from the equations (2.4.31) and (2.4.32) Substituting $y = l$:

$$\hat{Z}_{inp} = \frac{\hat{U}_2 \cosh \hat{\gamma} l + \hat{Z}_v \hat{I}_2 \sinh \hat{\gamma} l}{\frac{\hat{U}_2}{\hat{Z}_v} \sinh \hat{\gamma} l + \hat{Z}_v \hat{I}_2 \cosh \hat{\gamma} l} \quad (2.7.2)$$

After dividing the numerator and denominator by the current \hat{I}_2 and applying the relation $\hat{Z}_2 = \hat{U}_2/\hat{I}_2$, we obtain

$$\hat{Z}_{inp} = \hat{Z}_v \frac{\hat{Z}_2 \cosh \hat{\gamma} l + \hat{Z}_v \sinh \hat{\gamma} l}{\hat{Z}_2 \sinh \hat{\gamma} l + \hat{Z}_v \cosh \hat{\gamma} l} \quad (2.7.3)$$

Of the known values of voltage and impedance Z_i, Z_v st the input current can be determined

$$\hat{I}_1 = \frac{\hat{U}_i}{\hat{Z}_i + \hat{Z}_{inp}} \quad (2.7.4)$$

and the input voltage

$$\hat{U}_i = \hat{Z}_{inp} \hat{I}_1 \quad (2.7.5)$$

Voltage and current at loaded impedance \hat{Z}_2 is derived from equation (2.4.22) and (2.4.23) after substituting $x = l$:

$$\hat{U}_2 = \hat{U}_1 \cosh \hat{\gamma} l - \hat{Z}_v \hat{I}_1 \sinh \hat{\gamma} l \quad (2.7.6)$$

$$\hat{I}_2 = -\frac{\hat{U}_1}{\hat{Z}_v} \sinh \hat{\gamma} l + \hat{I}_1 \cosh \hat{\gamma} l \quad (2.7.7)$$

With matched transmission line, when $Z_1 = Z_v$ the fraction in relation (2.7.3) equals one and input impedance equal to wave impedance

$$Z_{inp} = Z_v \quad (2.7.8)$$

On transmission line propagates only successive wave and ratio of input/output voltages and currents

$$\frac{\hat{U}_1}{\hat{U}_2} = \frac{\hat{I}_1}{\hat{I}_2} = e^{\hat{\gamma} l} = e^{\hat{g}} \quad (2.7.9)$$

Trasmission rate

$$\hat{g} = \hat{\gamma} l = b + ja \quad (2.7.10)$$

Is defined by relation (2.7.9) characterize attenuation

$$b = \beta l \quad (2.7.11)$$

a phase shift

$$a = \alpha l \quad (2.7.12)$$

for the transmission line.

2.7.2 The distorted transmission line

Transmission line is often used for signal transmission which is more complicated than the harmonic voltage and current. These can be approximated by the sum of harmonic functions with different frequencies. Attenuation and specific shift are a function of frequency and the transfer of these signals occurs due to Attenuation differences of the harmonic components to the amplitude distortion and due to different phase velocity of these components to phase distortion.

Transmission line will be not distort when attenuation and phase velocity do not depend on frequency

$$\beta \neq f(\omega), \quad \nu_f \neq f(\omega) \quad (2.7.13)$$

Satisfies these requirements of transmission line, which primary parameters fulfill conditions

$$\frac{R_0}{L_0} = \frac{G_0}{C_0} \quad (2.7.14)$$

Substituting this condition into the definition of the relation and get treatment then attenuation

$$\beta = \sqrt{R_0 G_0} \quad (2.7.15)$$

and specific phase shift

$$\alpha = \omega \sqrt{L_0 C_0} \quad (2.7.16)$$

The phase velocity

$$\nu_f = \frac{\omega}{\alpha} = \frac{1}{\sqrt{L_0 C_0}} \quad (2.7.17)$$

and wave impedance

$$\hat{Z}_\nu = \sqrt{\frac{R_0}{G_0}} = \sqrt{\frac{L_0}{C_0}} \quad (2.7.18)$$

It doesn't depend on on the frequency and on transmission line propagate waves without changing shapes, they are attenuated

2.7.3 Frequency dependence of parameters in transmission line

Homogenous transmission line equations were derived assuming that primary parameters are constants. This assumption is limited to only a limited range of frequencies [23]. Because of the surface phenomenon, the resistance of the conductor increases with increasing frequency and inductances of transmission line is decreased. With increasing frequency, the loss in the dielectric is increasing too, which in turn will change in transverse conductivity. Frequency dependence of secondary parameters, evident from their definition is influenced by the frequency dependence of primary parameters. Frequency dependence is determined experimentally or by calculation using electromagnetic field theory. Example of depending frequency of specific attenuation and phase shift is shown in the Figure 2.13. Wave

impedance of a real transmission line usually has a capacitive character. The dependency of module and argument of wave impedance is illustrated in the Figure 2.14.

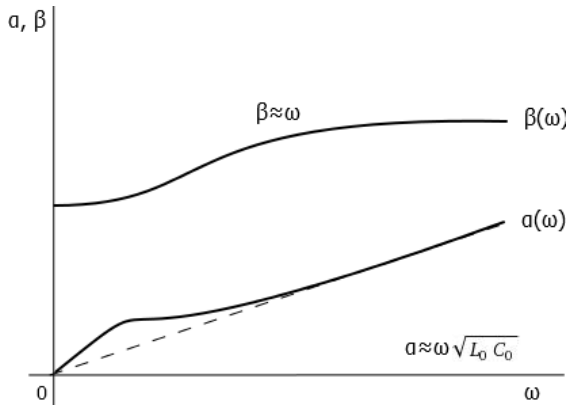


Figure 2.13: The dependency of frequency of specific attenuation and phase shift

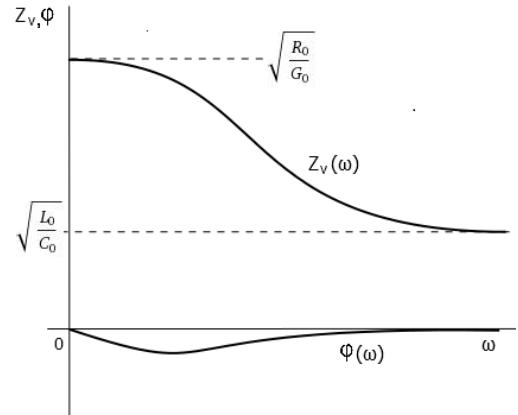


Figure 2.14: The dependency of module and argument of wave impedance

Usually the wave impedance for two-wire line is 200 to 800Ω, for cable lines (coaxial) are 50 to 100Ω. The phase velocity for aerial power lines is approximately equal to the speed of light c , for cable line is 0.4 to 0.8 c .

2.8 Analysis of transitional processes transmission line

Transients in transmission line are the result of the connection and disconnection of power supply, short circuits and line interruption, discontinuous shifts of load in transmission line [23]. For analysis of transitional processes the basic equation of transmission line is used

$$\begin{aligned} -\frac{\partial v}{\partial x} &= Ri + L \frac{\partial i}{\partial t} \\ -\frac{\partial i}{\partial x} &= Gv + C \frac{\partial v}{\partial t} \end{aligned}$$

derived in section 2.3.2. The solution can be obtain, when are known:

- *Boundary conditions*, i.e. voltage and current at the beginning and at the end of transmission line
- *Initial conditions*, which give the distribution of voltage and current in transmission line at time $t = 0$

Assuming the parameters R_0, G_0, C_0, L_0 are constants, a solution can simplify by using operators method. Voltage $u(x, t)$ and current $i(x, t)$ can be replaced by their images

$$U(x, p) = \int_0^\infty u(x, t) e^{-pt} dt \quad (2.8.1)$$

$$I(x, p) = \int_0^{\infty} i(x, t) e^{-pt} dt \quad (2.8.2)$$

where x is integration constant. By application of the Laplace transform for the basic equations of homogenous transmission line, we obtain

$$-\frac{\partial U(x, p)}{\partial x} = R_0 I(x, p) + pL_0 I(x, p) - L_0 i(x, 0) \quad (2.8.3)$$

$$-\frac{\partial I(x, p)}{\partial x} = G_0 U(x, p) + pC_0 U(x, p) - C_0 u(x, 0) \quad (2.8.4)$$

where $u(x, 0)$, $i(x, 0)$ are initial conditions. when they will be zero we obtain

$$-\frac{\partial U(x, p)}{\partial x} = (R_0 + pL_0) I(x, p) \quad (2.8.5)$$

$$-\frac{\partial I(x, p)}{\partial x} = (G_0 + pC_0) U(x, p) \quad (2.8.6)$$

which are After replacing complex variable p to variable $j\omega$ formally agree with the equations of harmonic steady state. Based on this analogy, we calculate

operator of propagation rate

$$\gamma(p) = \sqrt{(R_0 + pL_0)(G_0 + pC_0)} \quad (2.8.7)$$

and operator of wave impedance

$$Z_\nu(p) = \sqrt{\frac{(R_0 + pL_0)}{(G_0 + pC_0)}} \quad (2.8.8)$$

formally identical to the harmonic steady state will be solving equations after the known values of voltage $U_1(p)$ and current $I_1(p)$ at the beginning of the transmission line (Figure 2.15)

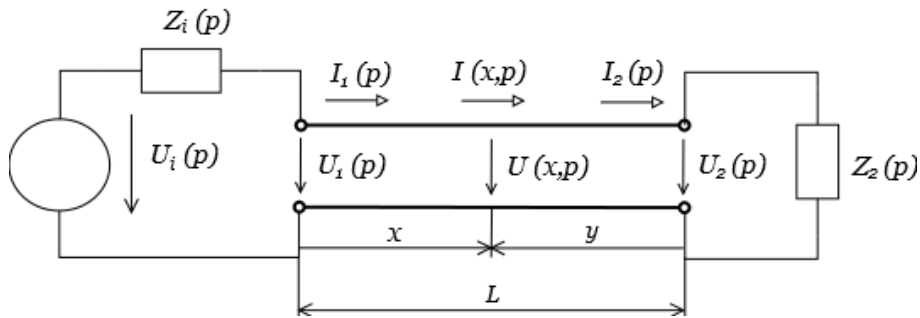


Figure 2.15

$$U(x, p) = U_1(p) \cosh \gamma(p) x - Z_\nu(p) I_1(p) \sinh \gamma(p) x \quad (2.8.9)$$

$$I(x, p) = -\frac{U_1(p)}{Z_\nu} \sinh \gamma(p) x + I_1(p) \cosh \gamma(p) x \quad (2.8.10)$$

for known value of voltage $U_1(p)$ and current $I_1(p)$ at the end of the transmission line valid that the voltage and current at a distance y from the end of the transmission line can be determined from equations

$$U(y, p) = U_2(p) \cosh \gamma(p) y - Z_\nu(p) I_2(p) \sinh \gamma(p) y \quad (2.8.11)$$

$$I(y, p) = \frac{U_2(p)}{Z_\nu} \sinh \gamma(p) y + I_2(p) \cosh \gamma(p) y \quad (2.8.12)$$

required voltage $u(x, t), u(y, t)$ and current $i(x, t), i(y, t)$ are obtained by inverse Laplace transform.

2.9 The pulse transfer on the real transmission line

Periodically pulsing signal propagates from the beginning of the transmission line can be expressed as the sum of the Fourier series of DC components, basic harmonic components and higher harmonic components[23]. from section 2.7 we know that the attenuation in real transmission line with frequency increases Figure 2.13. Increasing attenuation of higher harmonics components in the spectrum of the pulse. Forehead and rear of the pulse are deformed. As its illustrated in the Figure 2.16.

Forehead time and tulle of pulse are inversely proportional to the cutoff frequency f_m of transmission line, in which the attenuation begin to be increased according to the frequency

$$T_c \approx T_t \approx \frac{0.35}{f_m} \quad (2.9.1)$$

As a result of losses at transmission line the size of the pulse at the end of the transmission line decreases in comparison with the size of the pulse at the beginning of the transmission line.

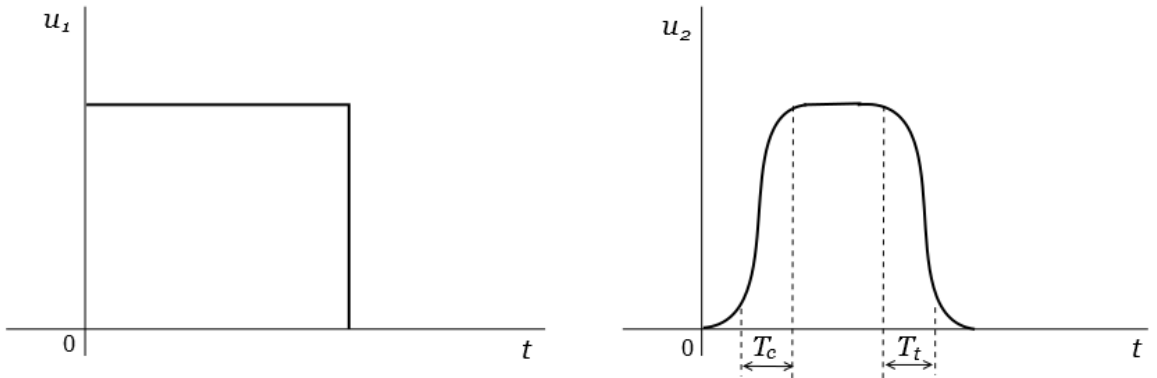


Figure 2.16

Chapter 3

Numerical and analytical methods for solving the differential equations

Now it's well known that the Telegrapher's equations (2.3.11) and (2.3.12) are pair of hyperbolic partial differential equations which describe transmission line in any location. So in this chapter will be shown the possible approaches to solving differential equations.

Differential equations are equations involving one or more functions and their derivatives.

This chapter is based on [28] [30] [26] [19] [5] [11] [21] [10] [27] [17] [8] [4].

3.1 Analytical solution

In differential equations there is usually a relationship between the function and its derivation [28]. For example for function $y(t) = A\sin(t)$ valids $y' = A\cos(t)$, $y'' = -A\sin(t)$. Then we can write

$$y''(t) + y(t) = 0 \quad (3.1.1)$$

For another function $y(t)$, As $yt = Be^t$ is $y'(t) = Be^t$. Then valid

$$y'(t) - y(t) = 0 \quad (3.1.2)$$

The differential equation represents a function, which can be substituted for the differential equation and obtain identity (equal results). Generally differential equations have infinitely many solutions but in practice usually some conditions are known (For example the initial), We are able to select just those that match the given situation. Analytical solution is shown in the example below

Example 3 Consider uniform linear motion of the body. From physics is known that the speed derivative distance by time, thus can be written as a differential equation $s'(t) = v(t)$ in this case the speed is constant then $v(t) = v$, is valid

$$s'(t) = v \quad (3.1.3)$$

Which corresponds to the differential equation for distance $s(t)$. By integration

$$s(t) = vt + C \quad (3.1.4)$$

where C is an arbitrary integral constant, which can be determined from initial conditions.

The result of analytical Solution is therefore a function of time and a particular value in the specific time is obtained by substituting this time into the resulting equation. Value can be determined at any point in which the function has solution.

Analytical methods are accurate, but can not be implemented by computer. There preferred the numerical solution to be used.

3.2 Numerical solution of Ordinary Differential Equations

Finding an analytical solution of large systems of differential equations is difficult, sometimes impossible. At present the computer technology is increasingly being used for numerical simulations. So, focus on the problem of numerical solution of ordinary differential equations with initial condition.

3.2.1 Basic terminology

The initial problem for ODE. Ordinary differential equation (ODE) is called the equation of n th order (ODR n) if the unknown function is a function of one variable and its highest derivative of the unknown function is the n th order. In this work the mainly differential equations of the first order is dealt. The *general shape* of the first order differential equations is

$$g(t, y(t), y'(t)) = 0. \quad (3.2.1)$$

Equation (3.2.1) can be written also explicitly

$$y'(t) = f(t, y(t)). \quad (3.2.2)$$

Such general solution of differential equation includes an integration constant that can have any value. Therefore, for the unambiguous determination of $y(t)$, the value of the function at determine point $t = t_0$ should be added, thus *initial condition*

$$y(t_0) = y_0. \quad (3.2.3)$$

Equation (3.2.2) together with the Initial condition (3.2.3) is called *Initial problem*, or *Cauchy problem*.

With the numerical solution of differential equations, the solution as continuous function is not looked for on the whole interval of solution $\langle a; b \rangle$, but we are looking for the values of the approximate solution only in a limited number of points which lying in the interval $(a = x_0 < x_1 < x_2 < \dots < x_n = b)$. Points x_1, x_2, \dots, x_n called **nodal** and their set is **network** [28]. a in our case is $y(x_0)$. difference $h_i = x_{i+1} - x_i$ is called a **step** in the network node x_i . If all the steps have a same length, then we are talking about **regular**

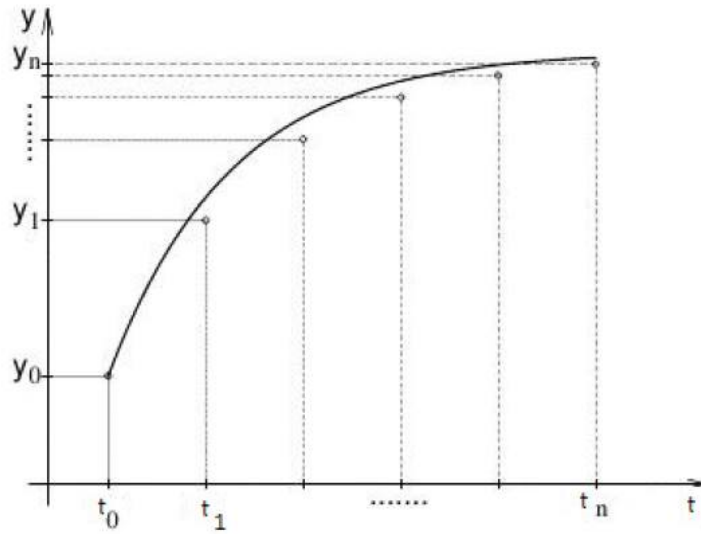


Figure 3.1: Exact and approximate solutions of differential equations.

network. The Figure 3.1 shows the exact solution of differential equation (solid line) and the approximate value indicated by the circles.

Generally, to solve these equations the numerical differentiation is used, which the function is replaced with interpolating polynomial or another approximations then differentiate approximating function. If the polynomial interpolation is used, then the derivative value of a function is replaced with an integration derivative value of a polynomial. If $P_n(x)$ the interpolation polynomial is given by a function $f(x)$ and nodes x_0, x_1, \dots, X_n , then

$$f'(x) = P'_n(x)$$

For higher order of the derivative (only to order n) then

$$f^{(s)}(x) = P_n^{(s)}(x)$$

Derivative value and polynomial interpolation might not match. As is shown in the Figure 3.1, the functional values of the nodal points are both in function and interpolation polynomial same, directive tangents to these two graphs (ie. the value of the first derivation) in nodal points are very different.

The simplest formula for the derivative of the first order can be obtained by a differentiation interpolation polynomial of the first degree of the nodes x_0 and $x_1 = x_0 + h$. If the function f has a second derivative on interval $\langle x_0; x_1 \rangle$, then there are points $\xi_0, \xi_1 \in \langle x_0; x_1 \rangle$ such that the following applies:

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{h} - \frac{h}{2} f''(\xi_0) \quad (3.2.4)$$

$$f'(x_1) = \frac{f(x_1) - f(x_0)}{h} - \frac{h}{2} f''(\xi_1) \quad (3.2.5)$$

These formulas can also be derived from the Taylor expansion of the function f . Graphically Equation (3.2.5) is represented as:

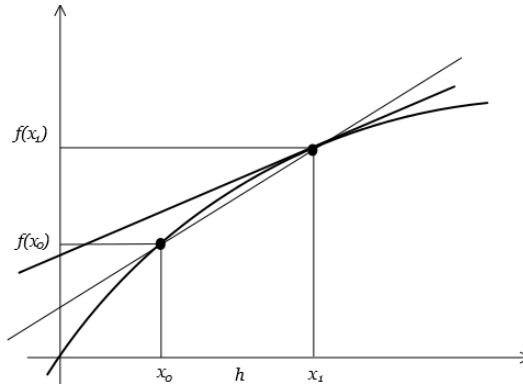


Figure 3.2: illustration of the equation (3.2.5)

By differentiating interpolation polynomial of the second stage of the nodes $x_0 = x_1 - h$, x_1 and $x_2 = x_1 + h$ we obtain a more accurate formula for the first derivative at these nodes. If the function f has a fourth derivative at the interval $\langle x_0, X_2 \rangle$ then there are points $\xi_0, \xi_1, \xi_2 \in \langle x_0, x_2 \rangle$ such that

$$f'(x_0) = \frac{-3f(x_0) + 4f(x_1) - f(x_2)}{2h} + \frac{h^2}{3} f'''(\xi_0) \quad (3.2.6)$$

$$f'(x_1) = \frac{f(x_2) - f(x_0)}{2h} - \frac{h^2}{6} f'''(\xi_1) \quad (3.2.7)$$

$$f'(x_2) = \frac{-3f(x_0) - 4f(x_1) + f(x_2)}{2h} + \frac{h^3}{3} f'''(\xi_2) \quad (3.2.8)$$

Using the second derivative of the same interpolation polynomial we get the formula for the second derivative of function f at point x_1 . Equation (3.2.7) is graphically illustrated as follows:

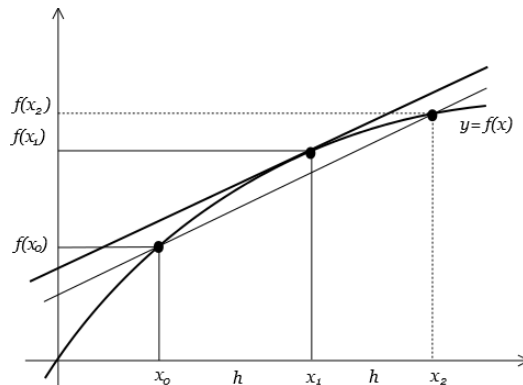


Figure 3.3: illustration equation (3.2.7)

The Figures 3.2 and 3.3 shows that, the value of the derivative of a function f at x_i is the directive tangents at this point (tangent in the Figures is drawn in black) is approximately equal to the Directive secant given points x_0 and x_1 , respectively, x_0 and x_2 (secant are drawn in gray).

3.3 Methods for the numerical solution of differential equations

Generally the numerical methods can be divided into two groups according to the number of the previous steps used to calculate approximately values in the next nodal point. Method using only information from one previous step is called **one-step method**. For example Euler's method and methods, which relied on it.

Method using information from many previous steps is called **multistep method** for example Adams-Bashforth methods.

3.3.1 Solution using Taylor polynomial

Is characterized by, that the approximate solution y_{i+1} in node t_{i+1} is calculated from the relationship, in which besides the unknown y_{i+1} there is the previous node t_i , the calculated value y_i in this node and of course the right side of differential equation $f(t, y)$. Formally, it can be written:

$$y_{i+1} = y_i + h\Phi(t_i, y_i, y_{i+1}, h, f). \quad (3.3.1)$$

Function Φ is a function of four variables t_i, y_i, y_{i+1} and h , it is dependent on the function $f(t, y)$. If the function Φ doesn't depend on y_{i+1} then we talking about explicit method, otherwise in case of dependence on y_{i+1} then the *implicit method* is investigated [26].

Consider the function $f(x)$, which has at the point $t = \eta$ derivatives up to the n -th order. Let us seek now polynomial $P(t)$ of degree n in the form of

$$P(t) = c_0 + c_1(t - \eta) + c_2(t - \eta)^2 + \dots + c_n(t - \eta)^n, \quad (3.3.2)$$

centered on the point η , such that the conditions

$$f^{(n)}(\eta) = P(\eta), f'(\eta) = P'(\eta), \dots, f^{(n)}(\eta) = P^{(n)}(\eta) \quad (3.3.3)$$

Expressing now higher derivative of a polynomial (3.3.2), in order to fulfill conditions in (3.3.3)

$$\begin{aligned} f(\eta) &= P(\eta) = [c_0 + c_1(t - \eta) + c_2(t - \eta)^2 + \dots + c_n(t - \eta)^n]_{t=\eta} = c_0, \\ f'(\eta) &= P'(\eta) = [c_1 + 2 \cdot c_2(t - \eta) + \dots + n \cdot c_n(t - \eta)^{n-1}]_{t=\eta} = c_1, \\ f''(\eta) &= P''(\eta) = [2 \cdot c_2 + \dots + n \cdot (n - 1) \cdot c_n(t - \eta)^{n-2}]_{t=\eta} = 2 \cdot c_2 \\ &\vdots \\ f^n(\eta) &= P^n(\eta) = [n \cdot (n - 1) \cdot \dots \cdot (n - (n - 1)) \cdot c_n]_{t=\eta} = n! \cdot c_n. \end{aligned}$$

The conditions in (3.3.3) leads to a system of equations

$$\begin{aligned} c_0 &= f(\eta), \\ c_1 &= f'(\eta), \\ c_2 &= \frac{f''(\eta)}{2 \cdot 1}, \\ &\vdots \\ c_n &= \frac{f^{(n)}(\eta)}{n!} \end{aligned} \quad (3.3.4)$$

After substituting the equations (3.3.4) into the original polynomial (3.3.2), the polynomial below is obtained

$$P(t) = f(\eta) + \frac{f'(\eta)}{1!}(t - \eta) + \frac{f''(\eta)}{2!}(t - \eta)^2 + \dots + \frac{f^{(n)}(\eta)}{n!}(t - \eta)^n, \quad (3.3.5)$$

which is known Taylor polynomial. Substituting into (3.3.5) $t = t_i + h = t_i + 1$ and $\eta = t_i$, known *explicit form* of Taylor polynomial is obtained

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2!}y''_i + \dots + \frac{h^n}{n!}y_i^{(n)} \quad (3.3.6)$$

Taylor polynomial (3.3.5) is the basis of all single-step numerical methods. It provides the most accurate approximation function. When calculating the solution is the need to take into account a larger number of members (at least of the order of tens). The problem, however, is getting higher derivatives. If Taylor's series method is well implemented, then it can be used very effectively for solving common problems.

For the solution of differential equations in this work the modern method of Taylor series is used by doc. Kunovský. By using this from accessing solves the differential equation system with TKSL, which will be described in next Chapter.

Different orders of the method (ODR) can be used during the calculation. For $ORD = 1$ therefore only the first member of the Taylor series is used:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

where h is the integrator step. in $ORD = 2$ the Taylor series is used to a second power step of h , so

$$y_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2!}f^{[1]}(t_n, y_n)$$

As mentioned, the method increases ORD automatically. This means that the values $\frac{h^p}{p!} f^{[p-1]}(t_n, y_n)$ are grossed up for increasing values of p until further adding members increase their accuracy.

The main precedence of this method is the automatic adjustment options of the number of orders of methods, it means Using many members of Taylor's series, to obtain the desired accuracy.

3.3.2 Euler's method

The Euler method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value [3]. It is the most basic explicit method for numerical integration of ordinary differential equations.

For the following initial-value problem

$$y' = f(t, y), t_0 \leq t \leq T \quad (3.3.7)$$

$$y(t_0) = y_0 \quad (3.3.8)$$

can be developed using Taylor series. We wish to approximate the solution at times $t_n, n = 1, 2, \dots$, where

$$t_n = t_0 + nh,$$

with h being a chosen *timestep*. Taking a Taylor expansion of the exact solution $y(t)$ at $t = t_{n+1}$ around the center t_n , we obtain

$$y(t_{n+1}) = y(t_n) + hy'(t_n) + \frac{h^2}{2}y''(\xi),$$

where $t_n < \xi < t_{n+1}$. Using the fact that $y' = f(t, y)$, we obtain a numerical scheme by truncating the Taylor series after the second term [21]. The result is a *difference equation*

$$y_{n+1} = y_n + hf(t_n, y_n),$$

where each y_n , for $n = 1, 2, \dots$, is an approximation of $y(t_n)$. This method is called *Euler's method*.

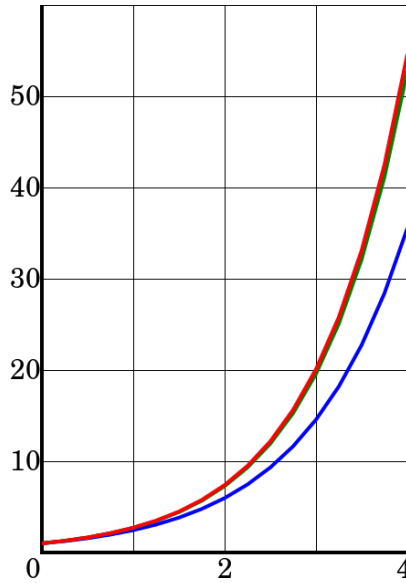


Figure 3.4: blue is the Euler method and red is the exact solution of $y = e^t$. The step size is $h = 1.0$

Differential equations using Euler's method Solution are not accurately, so in practice, it's not used. But there are two modification of this method, which increases its accuracy and very widely used group of Runge-Kutta methods, which ensues from it.

3.3.3 Runge-Kutta methods

There are two main reasons why Euler's method is not generally used in scientific computing. Firstly, the truncation error per step associated with this method is far larger than those associated with other, more advanced, methods (for a given value of h). Secondly, Euler's method is too prone to numerical instabilities.

Group of Runge-Kutta methods is one of the most important groups of one-step methods. The general shape of any methods of Runge-Kutta method is known as

$$y_{i+1} = y_n + h(w_1 k_1 + \dots + w_s k_s) \quad (3.3.9)$$

where

$$k_1 = f(x_n, y_n) \quad (3.3.10)$$

$$k_i = f\left(x_n + \alpha_i h, y_n + h \sum_{j=1}^{i-1} \beta_{ij} k_j\right) \quad i = 1, 2, \dots, s. \quad (3.3.11)$$

and w_i, α_i and β_{ij} constants chosen for method to have a maximum order.

The most widely known member of the Runge-Kutta family is generally referred to as „RK4“, „classical Runge-Kutta method“ or simply as „the Runge-Kutta method“.

Let an initial value problem be specified as follows.

$$y_0 = f(t, y), \quad y(t_0) = y_0.$$

Here, y is an unknown function (scalar or vector) of time t which we would like to approximate; we are told that y , the rate at which y changes, is a function of t and of y itself. At the initial time t_0 the corresponding y -value is y_0 . The function f and the data t_0, y_0 are given.

Now pick a step-size $h > 0$ and define

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\ t_{n+1} &= t_n + h \end{aligned}$$

for $n = 0, 1, 2, 3, \dots$, using

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right), \\ k_4 &= f(t_n + h, y_n + h k_3). \end{aligned}$$

Here y_{n+1} is the RK4 approximation of $y(t_{n+1})$, and the next value (y_{n+1}) is determined by the present value (y_n) plus the weighted average of four increments, where each increment is the product of the size of the interval, h , and an estimated slope specified by function f on the right-hand side of the differential equation.

Chapter 4

Conducting Experiments

This chapter is an introduction to a several systems, which may be used for solving differential equations: TKSL and MATLAB. Then, the comparison of the way that the systems solve differential equations will be discussed. TKSL uses Taylor polynomial, MATLAB has several integrated solvers.

This chapter is based on [16] [7] [20] [25] [15].

4.1 TKSL system

System TKSL (*Taylor-Kunovský Simulation Language*) is a simulation language and environment for computation of differential equations. All calculations are based on differential equations and are resolved exclusively using Taylor expansion (section 3.3.1). System allows the numerical solution of differential equations and is able to view the results as a graph [20]. Displaying a graph is one of the built elements in the system TKSL. Input or also input program is a set of differential equations. TKSL System was designed for MS-DOS, which nowadays can cause problems with compatibility. On the other side, it is necessary to appreciate its simplicity directness, and its hardware modesty. Now there is a new version of TKSL, called TKSL/C, which does not have some problems which has TKSL before. for purposes of this thesis is fully sufficient an old version of TKSL called TKSL/386.

4.2 Transmission line like an electrical circuit

In the previous chapters the primary line parameters were defined. These parameters largely correspond with the conventional electrical components: resistor (R), capacitor (C) and inductor (L). When the partial differential equations are transformed into the “regular” circuit, it may help with the understanding and analysis of the transmission line.

Primary parameters and relations derived from them depended on the distance from the voltage source and a length specific (analyzed) stretch. Transmission line can be splitted into parts of unit length and then analyze these parts.

Then it is possible to represent the whole transmission line as a groupe of two-ports. Length one of these two-ports is infinitesimal. Figure 2.4 showing a view from the most general elenment of trasmission line ,which can be after connection the voltage source and

ballast at the end (such as connecting a resistor) modeled in TKSL. In this case it was limited to one element (i.e. we could say that the whole transmission line was modeled with only one two-port) and the differential equations in TKSL can be defined as follows:

$$U'_{c1} = \frac{1}{C} i_{c1} \quad (4.2.1)$$

where $i_{c1} = (i_1 - i_2)$ then

$$U'_{c1} = \frac{1}{C} i_1 - i_2 \quad (4.2.2)$$

and

$$i'_2 = \frac{1}{L} U_L \quad (4.2.3)$$

where $U_L + U_{c2} - U_{c1} = 0$ and $U_L = U_{c1} - U_{c2}$ then

$$i'_2 = \frac{1}{L} (U_{c1} - U_{c2}) \quad (4.2.4)$$

The transmission line can be classified according to the first and the last element as a “capacitance is used at the first and the last elements” or “inductance is used at the first and the last elements” as below.

4.2.1 Electrical circuit using capacitance at the first and last elements

This class is presented by using capacitance at the first and the last elements as it is shown below:

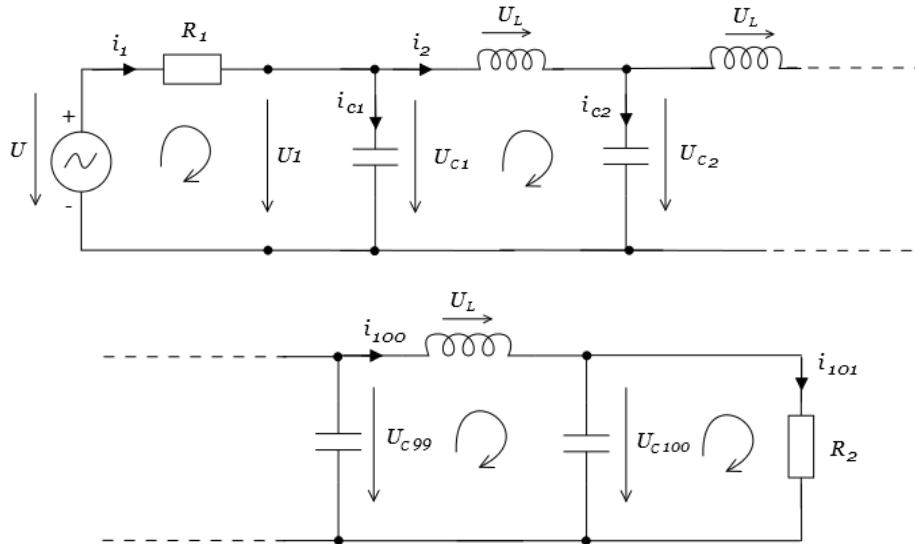


Figure 4.1: Transmission line beginning and ending with capacitance

Then for the first elements are valid

$$i_1 = \frac{(u - U_1)}{R_1}$$

$$U_1 = u_{c1}$$

$$u'_{c1} = \frac{1}{C} (i_1 - i_2)$$

$$i'_1 = \frac{1}{L} (u - u_{c1} - (R_1 * i_1))$$

$$u'_{c2} = \frac{1}{C} (i_2 - i_3)$$

$$i'_2 = \frac{1}{L} (u_{c1} - u_{c2})$$

and last elements

$$u'_{c99} = \frac{1}{C} (i_{99} - i_{100})$$

$$i'_{99} = \frac{1}{L} (u_{c98} - u_{c99})$$

$$u'_{c100} = \frac{1}{C} (i_{100} - i_{101})$$

$$i'_{100} = \frac{1}{L} (u_{c99} - u_{c100})$$

$$i_{101} = \frac{1}{R_2} u_{c100}$$

$$U_2 = u_{c100}$$

4.2.2 Electrical circuit using inductance at the first and last elements

Another class is presented by using inductance at the first and the last elements as it is shown below:

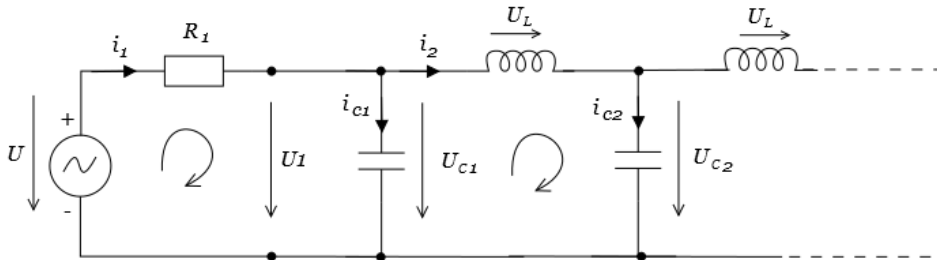


Figure 4.2: Transmission line beginning with inductance

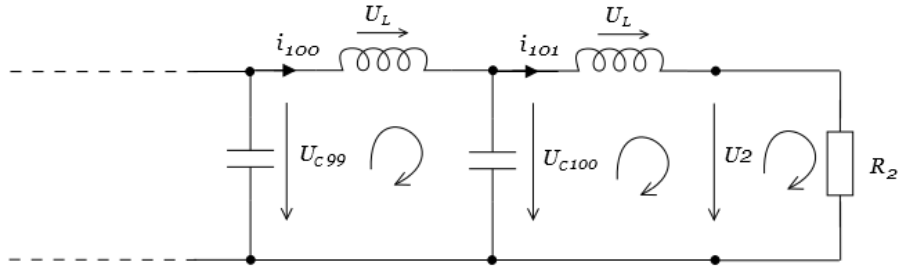


Figure 4.3: Transmission line ending with inductance

Then for the first element is valid

$$i_1 = \frac{(u - U_1)}{R_1}$$

$$U_1 = u - R_1 * i_1$$

$$u'_{c1} = \frac{1}{C} (i_1 - i_2)$$

$$i'_1 = \frac{1}{L} (u - u_{c1} - (R_1 * i_1))$$

$$u'_{c2} = \frac{1}{C} (i_2 - i_3)$$

$$i'_2 = \frac{1}{L} (u_{c1} - u_{c2})$$

and last elements

$$u'_{c99} = \frac{1}{C} (i_{99} - i_{100})$$

$$i'_{99} = \frac{1}{L} (u_{c98} - u_{c99})$$

$$u'_{c100} = \frac{1}{C} (i_{100} - i_{101})$$

$$i'_{100} = \frac{1}{L} (u_{c99} - u_{c100})$$

$$i_{101} = \frac{1}{L} (u_{c100} - u_2)$$

$$u_2 = i_{101} * R_2$$

After implementing this problem in our TKSL software to compare these situations we obtain a similar result(no changes) in output values voltage and current as it is shown in the Figure 4.4.

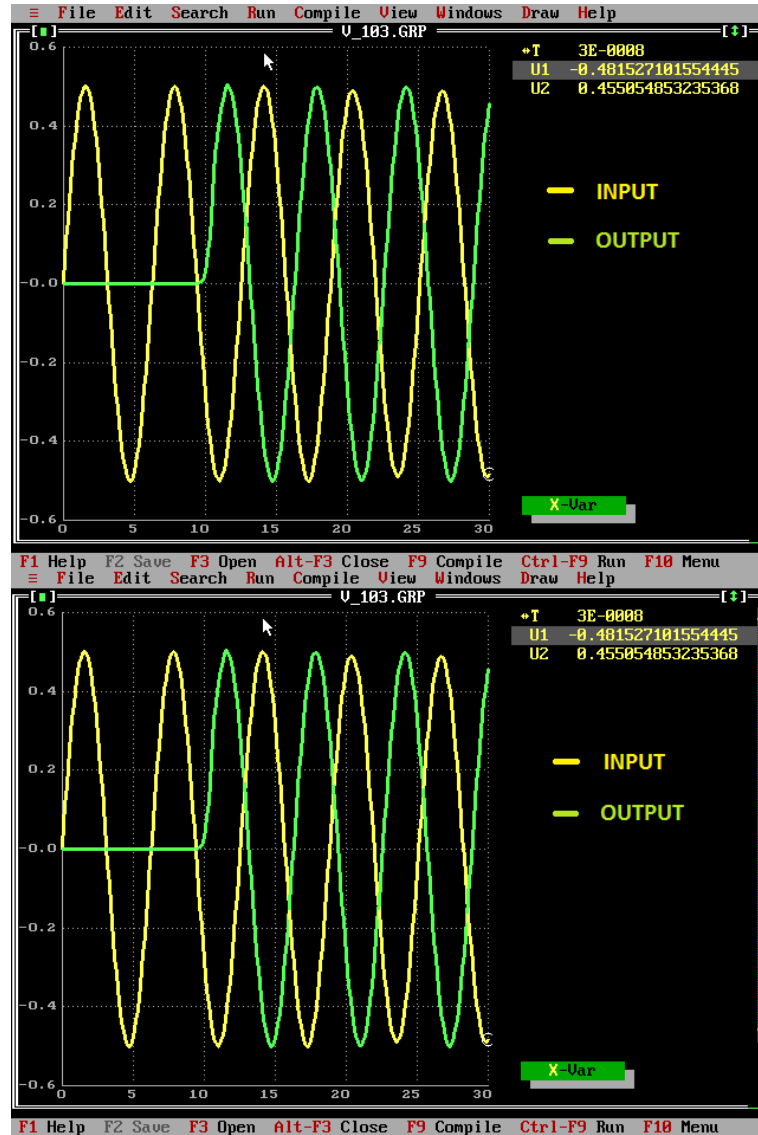


Figure 4.4: compare situations

4.3 Solving differential equations in TKSL/ 386 and TKSL/C

This section will be focus on the approach of solving the differential equations on TKSL. Differential equations intuitively can be written in TKSL/ 386 the same way as in TKSL/C. However, the „code“ written in TKSL/c is somehow simplified - eliminates the variable declarations and goes straight to the problem statement. Thus the program looks in the environment TKSL/386:

Algorithm 1: TKSL transmission line algorithm

```
Input: u1
Output: u2

var u,
      u1, u2;
      uc1,
      i1, i2,

const eps=1e-8, tmax=4.2e-8,
        dt=1e-10, L=1e-8,
        C=1e-12, R2=1e12,
        R1=50, om=3e9;

system
    { u=sin(om*t); }

    u1 = u-R1*i1;
    uc1' = 1/c*(i1-i2)    &z0;
    i1' = 1/l*(u-uc1-R1*i1)    &z0;
    :

sysend.
```

In TKSL/C the variable declarations part “**var**” is omitted and from the previous example remains only “**const**” section, but without the keyword and without constants “**tmax**”, “**eps**” and “**dt**”, which are entered as a parameter in a command line. Also, eliminates the system braces “**system**” and “**sysend.**” The syntax of differential equations does not differ.

4.4 MATLAB

The name MATLAB stands for MATrix LABoratory and it is a high-performance language for technical computing. It integrates computation, visualization, and programming environment. Furthermore, MATLAB is a modern programming language environment: it has sophisticated data structures, contains built-in editing and debugging tools, and supports object-oriented programming. These factors make MATLAB an excellent tool for teaching and research. MATLAB has many advantages compared to conventional computer languages (e.g., C, FORTRAN) for solving technical problems.

MATLAB is an interactive system whose basic data element is an array that does not require dimensioning. It has powerful built-in routines that enable a very wide variety of computations. It also has easy to use graphics commands that make the visualization of results immediately available. Specific applications are collected in packages referred to as toolbox. There are toolboxes for signal processing, symbolic computation, control theory, simulation, optimization, and several other fields of applied science and engineering.

Matlab is very useful for doing numerical computations with matrices and vectors. It can also display information graphically. The best way to learn what Matlab can do is to work through some examples at the computer.

MATLAB also allows implementation of algorithms, creation of user interfaces, and interfacing with programs written in other languages, including C, C++, Java, Fortran and Python.

The main reason for learning MATLAB in this thesis is so you can use it to analyze motion of an engineering system. To do this, you always need to solve a differential equation. MATLAB has powerful numerical methods to solve differential equations called ODE(ordinary differential equation). They are best illustrated by means of examples.

4.5 Solving differential equations in Matlab

This section will be focus on the approach of solving the differential equations in Matlab. Differential equations intuitively can be presented as:

$$\vec{y}' = A * \vec{y} + \vec{b} \quad (4.5.1)$$

where A is Jacobian matrix of constants and \vec{b} is a vector of constants, \vec{y} presents a vector of \vec{u}_C , \vec{i}_L and $\vec{y}(0) = 0$ as it is shown below:

$$\vec{y} = \begin{pmatrix} \vec{u}_C \\ \vec{i}_L \end{pmatrix} \quad (4.5.2)$$

and \vec{b} vector is presented as

$$\vec{b} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ \frac{1}{L} \sin(\omega t) \\ \vdots \\ \vec{0} \end{pmatrix} \quad (4.5.3)$$

and A is jacobian matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad (4.5.4)$$

The Jacobian is a matrix whose entries are first-order partial derivatives defined as where the function is given by m real-valued component functions, $y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n)$, continuous (smooth with no breaks or gaps) and differentiable (the derivative must exist at the point being evaluated). If $m = n$, then the Jacobian matrix is a square matrix [19].

Definition 1 Lets consider function $f_i(x_1, x_2, \dots, x_k)$ for $i = 1, 2, \dots, n$, which has partial derivatives $\frac{\partial f_i}{\partial x_j}$. Then Jacobian matrix \mathcal{J} is defined as

$$\mathcal{J} = \frac{df}{dx} = \left(\frac{\partial f}{\partial x_1} \quad \dots \quad \frac{\partial f}{\partial x_n} \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \quad (4.5.5)$$

and its determinant is called ($\det \mathcal{J}$)

$$\det \mathcal{J} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{vmatrix} = \left(\frac{\partial f_1}{\partial x_1} \dots \frac{\partial f_m}{\partial x_n} \right) - \left(\frac{\partial f_m}{\partial x_1} \dots \frac{\partial f_1}{\partial x_n} \right) \quad (4.5.6)$$

Example 4 Jacobian matrix - Consider the function $\mathbf{f}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f(x, y) = \begin{pmatrix} x^2 y \\ 5x + \sin y \end{pmatrix}.$$

Then we have

$$f_1(x, y) = x^2 y$$

and

$$f_2(x, y) = 5x + \sin y$$

and the Jacobian matrix of \mathbf{F} is

$$\mathcal{J}_{\mathbf{f}}(x, y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2xy & x^2 \\ 5 & \cos y \end{pmatrix}$$

As we mentioned that the jacobian matrix A contains 4 elements ($A_{11}, A_{12}, A_{21}, A_{22}$), each element is defined in Matlab as a sparse matrix with $nRLC * nRLC$ elements (where $nRLC$ is a number of elements in transmission line).

The vector \vec{b} is presented in Matlab as a zeros matrix of $2 * nRLC$ column and one row, which creates array of all zeros. Then the $\frac{1}{L} * u$ is assigned to the element $nRLC + 1$ (where u is the input signal and $nRLC + 1$ element presents differential equation i'_1 in transmission line)

MATLAB then in a cycle gradually increases indexes of a equations and splits into three parts, beginning of the transmission line, center of the transmission line and end of the transmission line. As it is shown below:

```

for i=1 to nRLC                                % nRLC number of T.L. elements

    if (i=1)                                       % first row

        y'_1 = u'_{c1} = \frac{1}{C} (i_1 - i_2)

        y'_{nRLC+1} = i'_1 = -\frac{1}{L} u_{c1} - \frac{R_1}{L} i_1 + b_{nRLC+1}

    elseif (i \neq 1) and (i \neq nRLC)

        y'_2 = u'_{c2} = \frac{1}{C} (i_1 - i_2)

        y'_{nRLC+2} = i'_2 = \frac{1}{L} (u_{c1} - u_{c2})

        \vdots

    elseif (i=nRLC)                               % last row

        y'_{nRLC} = u'_{nRLC} = -\frac{1}{RC} (u_{c_{nRLC}} + \frac{1}{C} i_{nRLC})

        y'_{2*nRLC} = i'_{nRLC} = \frac{1}{L} (u_{c_{nRLC-1}} - u_{c_{nRLC}})

    end
end

```

In this thesis the MATLAB code is investigated for the numerical solution of ODEs $y' = A * y + b$ with explicit Taylor series method as it is shown in section 3.3.1.

4.6 Using the Matlab ODE solvers

An ordinary differential equation (ODE) contains one or more derivatives of a dependent variable, y , with respect to a single independent variable, t , usually referred to as time. The notation used here for representing derivatives of y with respect to t is y' for a first derivative, y'' for a second derivative, and so on. The order of the ODE is equal to the highest-order derivative of y that appears in the equation.

The ODE solvers solve initial value problems with a variety of properties, they can work on stiff or nonstiff problems. Stiffness is a term that doesn't have a precise definition, but in general, stiffness occurs when there is a difference in scaling somewhere in the problem. For example, if an ODE has two solution components that vary on drastically different time scales, then the equation might be stiff. You can identify a problem as stiff if nonstiff solvers (such as ode45) are unable to solve the problem or are extremely slow. If you observe that a nonstiff solver is very slow, try using a stiff solver such as ode15s instead.

Table 4.1 provides general guidelines on when to use each of the different solvers [1].

Solver	Problem Type	Accuracy	When to Use
ode45	<i>Nonstiff</i>	Medium	Uses simultaneously fourth and fifth order RK formulas to make error estimates and adjust the time step accordingly. MATLAB recommends that ode45 is used as a first solver for a problem.
ode23		Low	Uses simultaneously second and third order Runge Kutta formulas to make estimates of the error, and calculate the time step size. Since the second and third order RK require less steps, ode23 is „ <i>less expensive</i> “, can be more efficient than ode45 at problems with crude tolerances, or in the presence of moderate stiffness.
ode113		Low to High	Uses variable-order Adams-Bashforth-Moulton solver. ode113 can be more efficient than ode45 at problems with stringent error tolerances, or when the ODE function is expensive to evaluate.
ode15s	<i>Stiff</i>	Low to Medium	Try ode15s when ode45 fails or is inefficient and you suspect that the problem is stiff.
ode23s		Low	ode23s can be more efficient than ode15s at problems with crude error tolerances. It can solve some stiff problems for which ode15s is not effective.
ode23t		Low	Use ode23t if the problem is only moderately stiff and you need a solution without numerical damping (the trapezoidal rule)
ode15i	<i>Fully implicit</i>	Low	ode15i is the only ODE solver that solves implicit equations

Table 4.1: Matlab ODE solvers

4.6.1 Solve Stiff ODEs

For some ODE problems, the step size taken by the solver is forced down to an unreasonably small level in comparison to the interval of integration, even in a region where the solution curve is smooth [2]. These step sizes can be so small that traversing a short time interval might require millions of evaluations. This can lead to the solver failing the integration, but even if it succeeds it will take a very long time to do.

Equations that cause this behavior in ODE solvers are said to be stiff. The problem that stiff ODEs pose is that explicit solvers (such as ode45) are untenably slow in achieving a solution. This is why ode45 is classified as a nonstiff solver along with ode23 and ode113.

Solvers that are designed for stiff ODEs, known as stiff solvers, typically do more work per step. The pay-off is that they are able to take much larger steps, and have improved numerical stability compared to the nonstiff solvers.

For most stiff problems, ode15s performs best. However, ode23s, ode23t, and ode23tb can be more efficient if the problem permits a crude error tolerance.

4.7 Experiments with homogeneous transmission line

In the previous section there were derived equations describing one element of the transmission line (two-port). Then two-port model was builded, which was then simulated in

instruments TKSL/386, TKSL/C and MATLAB. The following experiments will be on the homogenous Transmission line, ie. cascade of two-ports ,which was formed from the same elements and the same values.

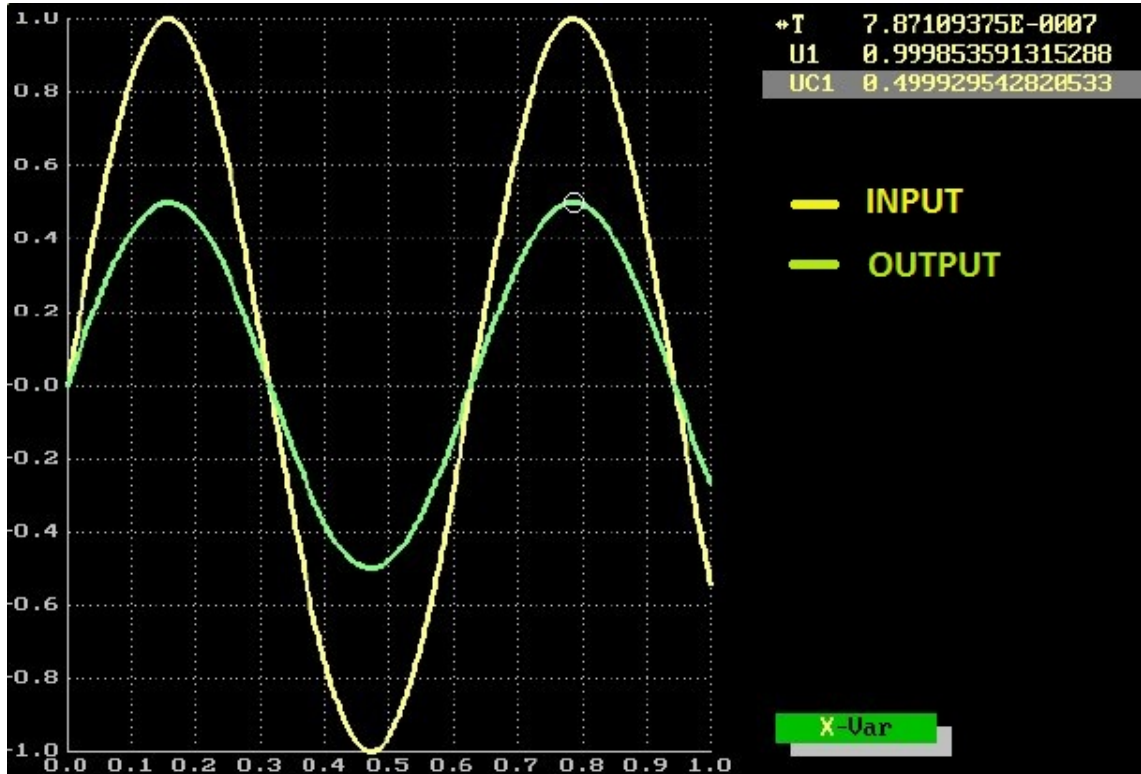


Figure 4.5: experiment one element in TKSL

First experiment was conducted with only one two-port, Its outputs is a graph as it is shown on the following Figure 4.5.

At the transmission line input was transferred a harmonic signal $U_m = \sin(\omega t)$ The graph shows that the voltage u_{c1} has against the applied voltage u lower amplitude, which meets the expectations.

Other experiments was performed with 10 two-ports modeled transmission line. In the graph, I have already launched an input voltage u and leaving only the voltage waveforms u_{c1} and u_{c10} . From the graph it is shown that these two waveforms are very similar and resulting values is different in one in a thousand to one in tenthousand (volts)

Therefore, I have also tried to increase the number of the element for transmission line formed from a cascade 20 elements i got a very similar results, so it was unremarkable

These experiments were performed in an environment TKSL/386, which has its own specific restrictions, for example the number of possible differential equations solved and the time needed for solving. Transmission line consists 100 elements (200 of differential equations), has the maximum number of elements which can TKSL/386 solve as it is shown in the

Figure 4.6. so i proceeded to use another software, which can solve more equations called “Matlab”

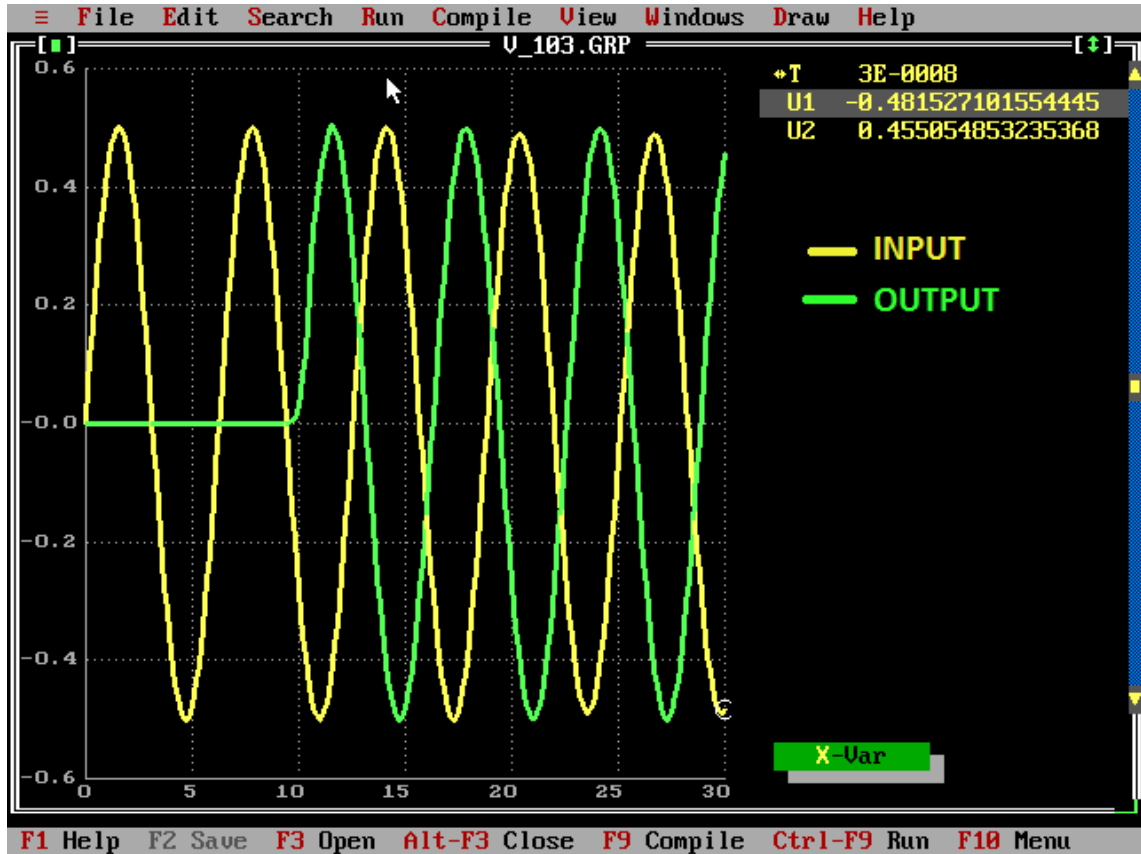


Figure 4.6: experiment 100 elements in TKSL

In MATLAB there were performed experiments on transmission line of at least 100 elements. The highest value of elements, which can be solved in MATLAB is infinite. Matrix and vectors, is implemented to generate this large systems of differential equations. output experiment of the transmission line with a 100 elements in Matlab is shown in the Figure 4.7

From Figures 4.6 and 4.7 is shown that the voltage U_{c100} or $U2$ is against voltage $U1$ delayed, This result is practically identical with the previous results experiment, which is already provided in system TKSL/386.

Harmonic signals takes some time before passing the whole transmission line elements, this time caused by the reaction of the used components (capacitance and inductance) on the connected harmonic signal.

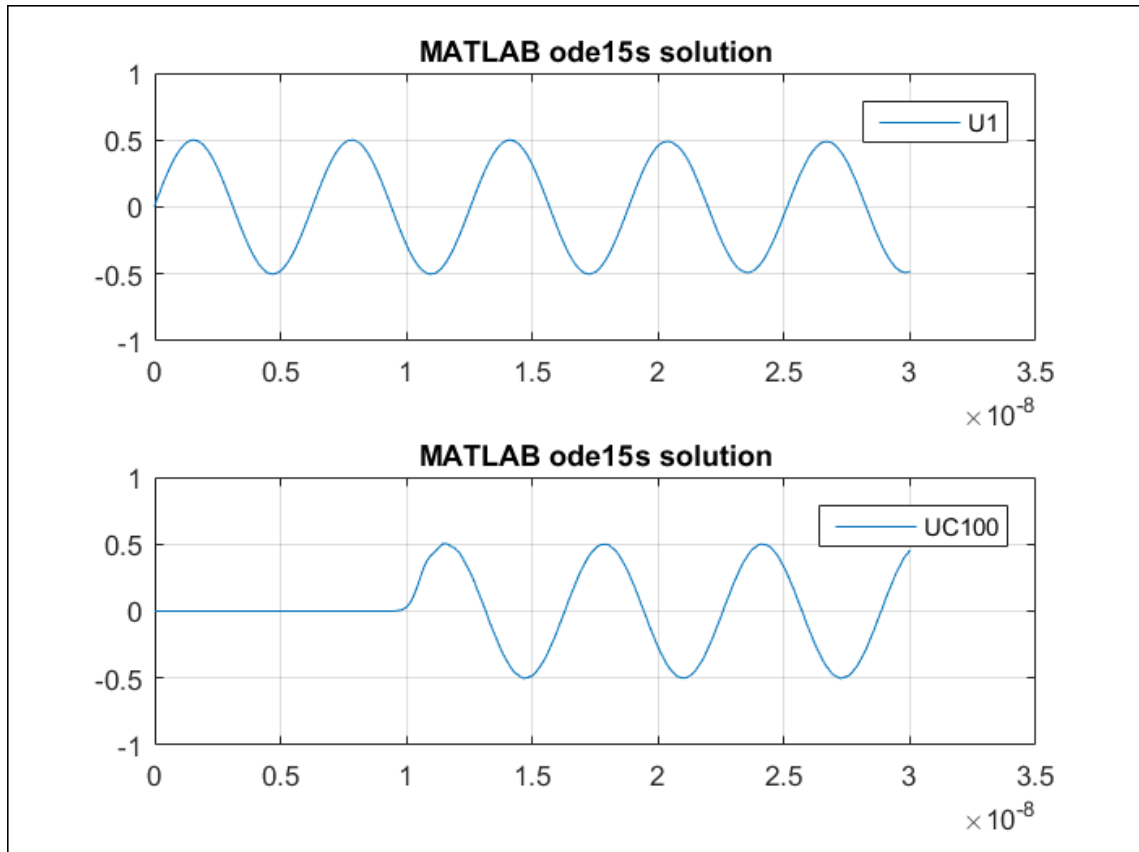


Figure 4.7: experiment 100 element in Matlab

Another examples with 1000 elements is implemented. transmission line evenly divided into 1000 parts of the same length, this length of each division is gradually approaching zero, then we can say that harmoniously signal passing these elements is with increasing distance from the source as it is shown in the Figure 4.9.

Results of experiments with 500 elements of homogeneous transmission line is shown in the Figure 4.8.

At the conclusion, we can say that the results of experiments with 1000 and 500 elements of homogeneous transmission line meets the expectations.

4.7.1 Length of the calculation

The length of the calculation comparison of numerical computation with the time-consuming in different solvers is shown on the table below 4.4.

Consider transmission line with these parameters:

$L=10^{-8}$; $R1=100$;
 $C=10^{-12}$; $R2=100$;
 $\omega = 1 \times 10^9$;

Solution methods trans. line length of transmission line	TKSL	ODE15s	ODE23	Taylor series
50	-	1.558729e+00	9.157196e-02	2.300273e-01
100	3e-0008	2.628126e+00	8.783597e-02	3.482762e-01
300	-	2.127624e+01	1.251556e-01	9.457567e-01
500	-	7.483213e+01	1.695998e-01	1.707365e+00
1 000	-	6.019412e+02	8.522765e-01	2.196717e+01
2 000	-	6.471619e+03	4.447670e+00	3.863677e+02

Table 4.2: Length of the calculation

Consider transmission line with these parameters:

L=10⁻¹¹; R1=100;
C=10⁻¹⁵; R2=100;

Solution methods trans. line length of transmission line	Frequency ω	ODE15s	ODE23	Taylor series
50	3*10 ⁹	6.217659e+01	5.740141e+00	6.072294e+01
100	3*10 ⁹	1.168535e+02	7.076802e+00	1.218671e+02
50	1*10 ¹¹	2.466492e+03	6.523801e+01	5.435500e+03
100	1*10 ¹¹	-	-	-

Table 4.3: Length of the calculation

Consider transmission line with these parameters:

L=10⁻⁸; R1=100;
C=10⁻¹²; R2=1e12;
om = 3*10⁹;

Solution methods trans. line length of transmission line	ODE15s	ODE23	Taylor series
50	1.019549e+00	3.562689e-03	2.449152e-01
100	1.874972e+00	2.539872e-03	3.378921e-01
300	1.108962e+01	2.243014e-03	9.432611e-01
500	3.062123e+01	2.447498e-03	1.692676e+00
1 000	1.279655e+02	8.4.941578e-03	3.906844e+00
2 000	1.524668e+03	6.049268e-03	4.514672e+01

Table 4.4: Length of the calculation

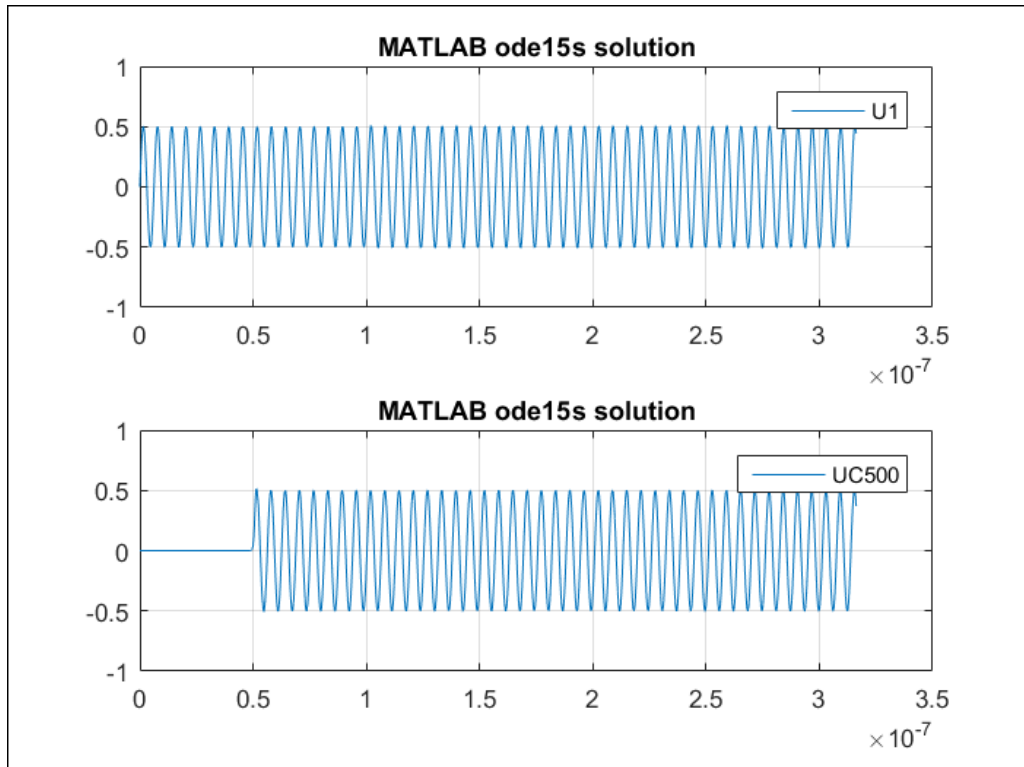


Figure 4.8: Experiment 500 element in Matlab

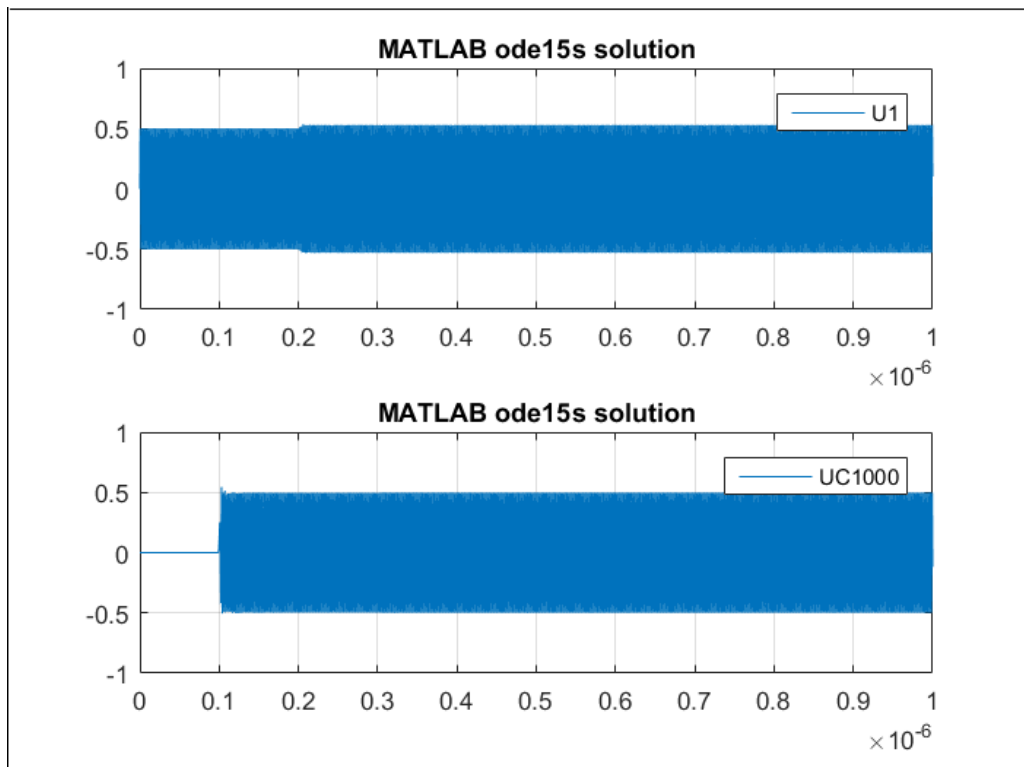


Figure 4.9: Experiment 1000 element in Matlab

Chapter 5

Conclusion

This thesis presents the standard methods of solution of telegraph equations, their circuit description by a system of differential equations, classification and they are mainly used for analysis of transient processes on homogeneous and non-homogeneous transmission line. The transmission line can be described by electrical circuit including an electrical components.

Using these skills the transmission line model was assembled, as a first the two-port transmission line is defined. The cascade of these two-ports is gradually created, which consisted for example 1 000 two-ports, simulation of such transmission line was prohibitively long and moreover, the transmission line with 500 two-ports has the same explanatory value for the analysis of transient processes, Therefore for forward experiments the 500 two-ports is used. Experimental results show interesting properties and changes which in the transmission line is accrued.

Main part of this thesis is focused on the analysis of the homogeneous and non-homogeneous transmission line with selected states and solving differential equations describing a special transmission line and time-consuming comparison of a numerical computation with the time-consuming at the other available literature.

Differential equations were solved in TKSL system and in MATLAB in which was implemented not only ODEs but also Taylor series for solving differential equations describing transmission line, where as a result was included a graphical solution given by these equations. It was already discussed suitability and applicability of the each evaluated method. In this thesis was also presented mathematical foundations on which the whole thesis is based.

Thesis assignment was fulfilled and the results of the thesis are presented in the previous chapters.

Appendix A

program for TKSL / 386 describing a transmission line element

Algorithm 2: 1

Input: u1

Output: u2

var u,

u1, u2,

uc1,

i1, i2;

const eps=1e-8, tmax=4.2e-8,

dt=1e-10, L=1e-8,

C=1e-12, R2=1e12,

R1=50, om=3e9;

system

{ u=sin(om*t); }

f=sin(om*t);

case t **of**

< 3.2e-9: u=f;

else u=0;

esac;

u1 = u-R1*i1;

uc1' = 1/c*(i1-i2) &0;

i1' = 1/l*(u-uc1-R1*i1) &0;

sysend.

Appendix B

program for TKSL / 386 describing transmission line with 10 elements

Algorithm 3: 1

```
var u,  
    u1, u2, uc1, uc2, uc3, uc4,  
    uc5, uc6, uc7, uc8, uc9, uc10,  
    i1, i2, i3, i4, i5, i6, i7, i8, i9, i10;  
  
const eps=1e-8, tmax=4.2e-8,  
    dt=1e-10, L=1e-8,  
    C=1e-12, R2=1e12,  
    R1=50, om=3e9;  
  
system  
    { u=sin(om*t); }  
    f=sin(om*t);  
  
    case t of  
    < 3.2e-9: u=f;  
    else u=0;  
    esac;  
  
    u1 = u-R1*i1;  
    uc1' = 1/c*(i1-i2)    &0;  
    i1' = 1/l*(u-uc1-R1*i1)    &0;  
  
    uc2' = 1/c*(i2-i3)    &0;  
    i2' = 1/l*(uc1-uc2)    &0;  
  
    uc3' = 1/c*(i3-i4)    &0;  
    i3' = 1/l*(uc2-uc3)    &0;
```

Algorithm 4: 1 (continuation)

```
uc4' = 1/c*(i4-i5)    &0;  
i4' = 1/l*(uc3-uc4)  &0;  
  
uc5' = 1/c*(i5-i6)    &0;  
i5' = 1/l*(uc4-uc5)  &0;  
  
uc6' = 1/c*(i6-i7)    &0;  
i6' = 1/l*(uc5-uc6)  &0;  
  
uc7' = 1/c*(i7-i8)    &0;  
i7' = 1/l*(uc6-uc7)  &0;  
  
uc8' = 1/c*(i8-i9)    &0;  
i8' = 1/l*(uc7-uc8)  &0;  
  
uc9' = 1/c*(i9-i10)   &0;  
i9' = 1/l*(uc8-uc9)  &0;  
  
uc10' = 1/c*(i10-i11) &0;  
i10' = 1/l*(uc9-uc10) &0;  
i11=1/R2*uc10;
```

sysend;

xaxis (0,8E-8);

yaxis (-2,2).

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